COMPLETELY MONOTONE FUNCTIONS AND SEQUENCES

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1. **Introduction.** Consider a function $f(x)$ defined by the Laplace-Stieltjes integral

$$f(x) = \int_0^\infty e^{-xt}\, dF(t), \tag{1}$$

where $F(t)$ is never decreasing and is bounded in every finite interval and the integral converges, say, for $x > 0$. In this paper we propose to deduce a simple inversion formula for (1) by an argument which enables us at the same time to prove in a very natural way some theorems of the theory of Laplace-Stieltjes integrals. Thus our argument provides an extremely simple proof of the well-known theorem to the effect that a function $f(x)$ can be represented in the form (1) if, and only if, it is completely monotonic, i.e., if it has for $x > 0$ derivatives of any finite order such that

$$(-1)^n f^{(n)}(x) \geqq 0 \qquad (n = 0, 1, \cdots). \tag{2}$$

A theorem which is substantially equivalent to this statement is proved by F. Hausdorff.[1] In its present form it was first formulated by S. Bernstein,[2] and subsequently independently by D. V. Widder.[3] A simplified proof was then given by J. D. Tamarkin,[4] and subsequently Widder himself proposed an alternative proof.[5] Widder has also proposed some inversion formulas[6] for integrals of type (1), but the formula given in the sequel seems nevertheless to be of some interest and proves the theorem of Hausdorff-Bernstein in a more direct way. Moreover, the method is easily applicable to other problems, in particular to the problems treated by Widder in his papers referred to above. As an example we deduce a necessary and sufficient condition that a function $f(x)$ be representable in the form (1), when $F(t)$ is only supposed to be of bounded variation in every finite interval; this condition is equivalent to a similar condition obtained by Widder.[7] Also the case of a function $F(t)$ with a bounded derivative will be treated in the sequel.

We then deal with the general interpolation problem for completely mono-

Received April 1, 1939.

[1] Hausdorff [4], II, Theorem 3. (Numbers in brackets refer to the list of references at the end of the paper.)

[2] Bernstein [1].

[3] Widder [10].

[4] Tamarkin [9].

[5] Widder [11], Theorems 17–18.

[6] Widder [11].

[7] Widder [10], Theorem 12.