THE WEIERSTRASS CONDITION FOR MULTIPLE INTEGRAL VARIATION PROBLEMS

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Consider the problem of minimizing a multiple integral

$$I = \int_R f(t, x, p)\, dt = \int \cdots \int f(t, x, p)\, dt_1 \cdots dt_n$$

in a class of admissible manifolds in (t, x)-space with equations in the form $x = x(t)$, where $x = (x^1, \cdots, x^m)$, $t = (t_1, \cdots, t_n)$, and p denotes the matrix $(p^i_\alpha) = (\partial x^i / \partial t_\alpha)$. The Weierstrass E-function has the form

$$E(t, x, p, P) = f(t, x, P) - f(t, x, p) - (P^i_\alpha - p^i_\alpha) f^\alpha_i(t, x, p),$$

where $f^\alpha_i = \partial f / \partial p^i_\alpha$ and the usual summation convention is used. We suppose that f and its partial derivatives f^α_i are continuous in a certain region S of (t, x)-space for all p.[1] For the class of admissible manifolds we take all manifolds $x = x(t)$ lying in S, of class D′ on the fixed region R of t-space, and having a fixed boundary over the boundary of R. Then *a necessary condition for a minimum of I is that $E(t, x, p, P) \geqq 0$ for all (t, x, p) on the minimizing manifold and for all P such that the matrix $P - p = (P^i_\alpha - p^i_\alpha)$ has rank one.*

A very brief and elementary proof for the condition is given in §1 below.[2] The proof is similar to one given by the author for simple integral problems.[3] The Weierstrass condition as stated obviously applies also to problems in parametric form. In §2 the condition is transformed so as to be expressed entirely in terms of the n-rowed minors of the matrices p and P.[4]

1. **Proof of the Weierstrass condition.** Let the minimizing manifold M_0 have equations $x^i = \phi^i(t)$, and suppose that the partial derivatives $\partial \phi^i / \partial t_\alpha$ are continuous near $t = \bar{t}$. Let these derivatives for $t = \bar{t}$ be denoted by p^i_α. If the matrix $\Delta p = P - p$ has rank one, it may be represented in the form $\Delta p^i_\alpha =$

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[1] Obviously we could also consider restrictions on the admissible values of the p^i_α.

[2] For more general classes of admissible manifolds McShane has shown that $E \geqq 0$ almost everywhere. See Annals of Mathematics, vol. 32(1931), pp. 578–590. Another type of proof has been developed by Coral. See this Journal, vol. 3(1937), pp. 585–592.

[3] *A proof of the Weierstrass condition in the calculus of variations*, American Mathematical Monthly, vol. 41(1934), pp. 502–504.

[4] In the cases $n = 1$ and $n = m - 1$ (with more general hypotheses on the class of admissible manifolds) McShane gave a direct proof of the transformed condition for parametric problems. See Annals of Mathematics, vol. 32(1931), pp. 723–733.