A MEAN ERGODIC THEOREM

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In this note we shall give an ergodic theorem of the von Neumann type[1] for continuous n-parameter uniformly bounded semi-groups of linear transformations in a Banach space. The method of proof is an extension of one used by F. Riesz[2] and K. Yosida[3] for the discrete case in one dimension.

Throughout the note we shall use the following notation and terminology. E_n is Euclidean n-space, E_n^+ is the set of points $\alpha = (a_1, \cdots, a_n)$ in E_n with $a_i \geqq 0$ $(i = 1, \cdots, n)$, I_r is a cube of side $r > 0$ in E_n. That is, I_r is the set of points $\alpha = (a_1, \cdots, a_n) \in E_n$ for which

$$c_r^j \leqq a_j \leqq c_r^j + r \qquad (j = 1, \cdots, n), \tag{1}$$

where c_r^j $(j = 1, \cdots, n)$ is an arbitrary real function defined for $r > 0$. When we are dealing with functions defined only on E_n^+, it will be assumed that all cubes mentioned are in E_n^+. We shall sometimes use the symbol V_r for the volume r^n of a cube I_r. A set T_α $(\alpha \in E_n^+)$ of linear transformations in a Banach space $\mathfrak{B}$ is said to form a *semi-group* in case

$$T_{\alpha+\beta} = T_\alpha T_\beta, \qquad \alpha, \beta \in E_n^+, \tag{2}$$

and in case T_α is defined for every $\alpha \in E_n$, it is said to form a *group* if T_0 is the identity and equation (2) holds for all $\alpha, \beta \in E_n$. A group (or semi-group) T_α is said to be *uniformly bounded* in case $\| T_\alpha \|$ is bounded in α, and it is said to be *weakly measurable* in case the numerical function $\bar{x}T_\alpha x$ is measurable (in the sense of Lebesgue) for every $x \in \mathfrak{B}$ and $\bar{x} \in \bar{\mathfrak{B}}$ (the space conjugate to $\mathfrak{B}$). A function x_α defined on E_n (or E_n^+) is said to be *almost separably valued* in case there is a set E_0 of measure zero such that the set of points x_α, $\alpha \bar{\in} E_0$, is a separable subset of $\mathfrak{B}$. If for every cube I_r of side $r > 0$ we have a point $y(I_r) \in \mathfrak{B}$, then the set of all $y(I_r)$ is said to be *weakly compact with respect to* $r \to \infty$ in case every particular set I_r $(0 < r < \infty)$ of cubes contains a sequence I_{r_p} with $r_p \to \infty$ and $y(I_{r_p})$ converging weakly to an element of $\mathfrak{B}$. A function x_α defined on a cube is said to be *measurable* in case it is the limit almost every-

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[1] J. von Neumann, *Proof of the quasi ergodic hypothesis*, Proc. Nat. Acad., vol. 18(1932), pp. 70–82.

[2] F. Riesz, *Some mean ergodic theorems*, Jour. London Math. Soc., vol. 13(1938), pp. 274–278.

[3] K. Yosida, *Mean ergodic theorem in Banach spaces*, Proc. Imp. Acad. Tokyo, vol. 14(1938), pp. 292–294.