PSEUDO-NORMED LINEAR SPACES AND ABELIAN GROUPS

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In his study of linear topological spaces[1] J. von Neumann has introduced the notion of a pseudo-metric, which can be defined in any locally convex[2] linear topological space. It is a real-valued function and has many of the properties of a norm, including the fulfillment of the triangular inequality. The peculiarity of this pseudo-metric is that its value depends not only on the element x of the space but also on a neighborhood U out of a system of neighborhoods of the origin. In the present paper the pseudo-metric is generalized in the following way. First, the triangular inequality is replaced by a much weaker condition of continuity, and secondly, the neighborhood system is replaced by any "strongly" partially ordered set, and this makes it possible to postulate a "pseudo-norm" for a linear space without first postulating a topology.

In the first section of the paper pseudo-normed linear spaces are defined and shown to be the same as linear topological spaces. The pseudo-norm includes as special cases both the pseudo-metric of von Neumann and the pseudo-norm previously defined by the author in [3].

Pseudo-normed Abelian groups are defined in the second section of the paper, and it is shown that a topological Abelian group G has a "linear topological extension" if and only if the topology of G can be generated by a pseudo-norm. Thus pseudo-normed Abelian groups are just those Abelian groups which are topological subgroups of a linear topological space.

1. Let D be a set with elements $a, b, c, \cdots$ which has the composition property of Moore and Smith. That is, we are postulating that (i) if $a > b$, then $b \not> a$; (ii) if $a > b$ and $b > c$, then $a > c$; (iii) given a and b there exists c such that $c \geqq a$ and $c \geqq b$. The set D together with the relation $>$ will be called a *strongly partially ordered space.*[3]

A linear space L will be said to be pseudo-normed with respect to D if there exists a real-valued function $n(x, d)$ defined on LD which satisfies the following postulates:

(1) $n(x, d) \geqq 0$; $n(x, d) = 0$ for all $d \in D$ implies $x = \theta$, where θ represents the zero of L;

(2) $n(\alpha x, d) = | \alpha | n(x, d)$ for all real α;

Received March 5, 1939; presented to the American Mathematical Society, September 6, 1938.

[1] See [5]. The numbers in brackets refer to the bibliography at the end of the paper.

[2] Local convexity is equivalent to von Neumann's convexity. See p. 158 of [6].

[3] Such sets were first used in general topology by G. Birkhoff. See [2].