ANNIHILATOR IDEALS AND REPRESENTATION ITERATION FOR ABSTRACT RINGS

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1. **Introduction.** It has been shown[1] that in a ring Γ_0 with additive group M_0, every element g acting as a left multiplier on Γ_0, defines an operator λ in the endomorphism ring Ω_0 of M_0, such that $g\Gamma_0 = \lambda\Gamma_0$, $g\Gamma_0$ being the ordinary ring multiplication, and $\lambda\Gamma_0$ the map of M_0 by the operator λ. The set of all such operators λ forms a subring $\mathfrak{L} = \mathfrak{L}(\Gamma_0)$ of Ω_0 to which Γ_0 is ring homomorphic. We shall call $\mathfrak{L}$ the *left representation* of Γ_0. Similarly the right multipliers define a *right representation* $\mathfrak{R}^T$ of Γ_0, where $\mathfrak{R}^T$ consists of a ring inverse isomorphic to the subring $\mathfrak{R}$ of operators of Ω_0 defined by the right multipliers; $\mathfrak{R}^T$ being used in order that Γ_0 be ring homomorphic to $\mathfrak{R}^T$ in the ordinary sense. We have new rings $\mathfrak{L}$, $\mathfrak{R}^T$, each having an additive group with corresponding operator rings. We are thus free to iterate the process, forming left or right representations of representations in any order. We shall study these representation rings and their isomorphism to residue class rings of Γ_0 modulo certain annihilating ideals.[2]

2. **The ideal theory.** Let Γ be an abstract ring, and define Σ^m as the set of all products $s_1 \cdots s_m$ of m factors, $s_i \in \Sigma \subset \Gamma$. Denote by (r, l) the set of all x of Γ such that $\Gamma^r x \Gamma^l = 0$ $(r, l = 0, 1, \cdots)$. Γ^0 is merely deleted wherever it occurs formally. Thus $(0, 0) = 0$, and $(0, 1)\Gamma = 0$. It is clear that (r, l) is a 2-sided ideal in Γ and $(r, l) \subset (r + s, l + k)$ for all s, k. If Δ is a 2-sided ideal in Γ, we write $\Gamma - \Delta$ for the residue class ring of Γ mod Δ.

Let λ be the least l for which $(0, l) = (0, l + 1)$, ρ the least r such that $(r, 0) = (r + 1, 0)$. Γ is said to be of *l-type* λ, of *r-type* ρ. If $\lambda = 0$, Γ is called *l-definite*; if $\rho = 0$, *r-definite*; and if both, *definite*. Conditions on Γ for finiteness of type are given in §3. We shall consider only rings with finite ρ, λ.

THEOREM 1. *If in* $\Gamma-(r, l)$, $(s, k) = 0$, *then in* $\Gamma-(r - i, l - j)$, $(i, j) = (i + s, j + k)$, *and conversely.*

Suppose $\Gamma^{i+s}x\Gamma^{j+k} \equiv 0$ in $\Gamma-(r - i, l - j)$; i.e., $\Gamma^{r+s}x\Gamma^{l+k} = 0$ in Γ and $\Gamma^s x\Gamma^k \equiv 0$ in $\Gamma-(r, l)$. Hence $x \equiv 0$ in the latter ring, and $\Gamma^r x \Gamma^l = 0$ in Γ. Hence in $\Gamma - (r - i, l - j)$, $\Gamma^i x \Gamma^j \equiv 0$, and in this ring $(i + s, j + k) \subset (i, j)$. The argument is reversible.

Received February 24, 1939.

[1] *Rings as groups with operators*, Bull. Amer. Math. Soc., vol. 45(1939), pp. 274-279.

[2] The extension of the theory to rings with rings of operators is readily made. Thus in the case of linear associative algebras $\mathfrak{L}$ and $\mathfrak{R}^T$ are the usual left and right regular matrix representations.