SOME PROPERTIES OF POLYNOMIAL SETS OF TYPE ZERO

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1. **Introduction.** Pincherle,[1] in his study of the difference equation

$$\sum_{n=1}^{k} c_n \phi(x + h_n) = f(x),$$

was led to consider a set of Appell polynomials, in infinite series of which solutions could be represented. We considered the same equation[2] by means of a different Appell set, the change resulting in a significant alteration of the regions of convergence (for the series). This permitted an enlargement of the class of functions $f(x)$ for which a solution could be shown to exist. Recently we treated the more general equation[3] (linear differential equation of infinite order)

$$L[y] \equiv a_0 y + a_1 y' + \cdots = f(x),$$

where, under suitable conditions on L and f, a solution was found. Here, too, it was possible to relate the equation to a corresponding problem of expanding functions in series of Appell polynomials. It is this close relation to functional equations that adds interest to the study of Appell sets.

As is well known, Appell sets $\{P_n(x)\}$ $(n = 0, 1, \cdots)$ are characterized by either of the equivalent conditions

$$P_n'(x) = P_{n-1}(x) \qquad (P_n \text{ a polynomial of degree } n); \tag{1.1}$$

$$A(t)e^{tx} \cong \sum_{0}^{\infty} P_n(x)t^n, \tag{1.2}$$

where $A(t) \cong \sum a_n t^n$ is a formal power series, and where the product on the left of (1.2) is formally expanded in a power series in accordance with the Cauchy rule. We shall say that the series $A(t)$ is the *determining series* for the set $\{P_n\}$.

For the particular equation

$$y(x + 1) - y(x) = f(x),$$

Pincherle used the Appell set with $A(t) = 1/(e^t - 1)$, getting essentially the Bernoulli polynomials. We used $A(t) = e^t - 1$, so that $n!P_n(x) = (x + 1)^n - x^n$. Now this equation is also associated with the important set of Newton polynomials

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[1] Acta Mathematica, vol. 48(1926), pp. 279–304.

[2] Trans. Amer. Math. Soc., vol. 39(1936), pp. 345–379, and vol. 41(1937), pp. 153–159.

[3] This Journal, vol. 3(1937), pp. 593–609.