CERTAIN CONGRUENCES INVOLVING THE BERNOULLI NUMBERS

By H. S. Vandiver

In a previous paper[1] I gave a general theorem concerning congruences in rings which yields congruences involving Bernoulli numbers, in particular, the relation known as Kummer's Congruence:[2]

$$h^n(h^{p-1} - 1)^j \equiv 0 \pmod{p^j} \tag{1}$$

for $n - 1 \geqq j$, $n \not\equiv 0 \pmod{p-1}$, where p is an odd prime, the left member is expanded in full by the binomial theorem and then b_t/t is substituted for h^t. The b's are defined by the recursion formula

$$(b + 1)^n = b_n \qquad (n > 1),$$

where we expand the left member by the binomial theorem and substitute b_k for b^k. The b's give the Bernoulli numbers.

A number of proofs of (1) have been given, all, as far as I know, including the restriction $n \not\equiv 0 \pmod{p-1}$; in fact, a simple inspection shows that the result does not hold when $n \equiv 0$. The question naturally arises whether there is some complementary theorem which provides for the case $n \equiv 0 \pmod{p-1}$. Nielsen[3] investigated this problem and found the relation

$$1 - \frac{1}{p} \equiv \sum_{s=1}^{s=r} (-1)^{s+s\mu} \binom{r}{s} B_{\mu s} \pmod{p},$$

$$B_n = (-1)^{n-1} b_{2n}; \qquad \mu = \tfrac{1}{2}(p-1).$$

To obtain this he sets

$$S_n(m) = 1^n + 2^n + \cdots + (m-1)^n$$

and employs the following relation (proved by him previously),

$$\sum_{k=0}^{k=r} (-1)^k \binom{r}{k} S_{2n+2k\mu}(p) \equiv \sum_{s=1}^{p-1} S^{2n}(1 - S^{2\mu})^r,$$

and

$$S_{2m}(p) \equiv (-1)^{m-1} B_m p \pmod{p^2} \qquad (p > 3,\ m > 1).$$

The result follows. Now it is not clear how this argument may be extended to the examination of (2), modulo p^α, in lieu of modulo p. But the method of my

Received January 30, 1939.

[1] Bull. Amer. Math. Soc., vol. 43(1937), pp. 417–423.

[2] Journal für die Mathematik, vol. 41(1851), pp. 368–372.

[3] *Traité des Nombres de Bernoulli*, pp. 277–278.