THE ALGEBRA OF LATTICE FUNCTIONS

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I. Introduction

1. The numerous disconnected results on numerical functions (that is, functions on the positive integers to the complex numbers) which are summarized in the first volume of Dickson's *History* have been welded into a simple and coherent theory by Bell in a series of papers culminating in his *Algebraic Arithmetic* (Bell [1][1]). Bell has shown in detail (see, for example, Bell [2], [3], [4], [5], [6]) that all the various inversion formulas, factorability properties, numerical integrations, and so on, of these functions follow from three basic facts.

I. *The set of all numerical functions form a ring with respect to the operations of addition and Dirichlet multiplication.*

The sum $\sigma = \phi + \psi$ of two numerical functions ϕ and ψ is defined by $\sigma(n) = \phi(n) + \psi(n)$, while their Dirichlet product $\pi = \phi\psi$ is defined by

$$\pi(n) = \sum_{d\delta = n} \phi(d)\psi(\delta). \tag{1.1}$$

II. *The set of all numerical functions ϕ such that $\phi(1) \neq 0$ form a group with respect to Dirichlet multiplication.*

The inverse ϕ^{-1} of ϕ satisfies

$$\sum_{d\delta = n} \phi(d)\phi^{-1}(\delta) = \begin{cases} 1 & \text{if} \quad n = 1; \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

For example, the inverse of the function ζ defined by $\zeta(n) = 1$ for all n is the Möbius function $\mu(n)$.

III. *The set of all factorable functions is closed with respect to the operation of Dirichlet multiplication.*

A function ψ is said to be factorable if

$$\psi(mn) = \psi(m)\psi(n) \qquad \text{if } m, n \text{ are co-prime.} \tag{1.3}$$

It may be shown that the factorable functions form a group with respect to Dirichlet multiplication, on excluding the trivial function ω vanishing for all integers n.

Since the positive integers form a semi-ordered set with respect to the relation x divides y, and indeed a lattice, it is natural to ask whether results of like simplicity and generality hold for functions on semi-ordered sets and lattices. But since both Dirichlet multiplication and factorability depend upon a

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[1] Numbers in brackets refer to the references at the end of the paper.