COMMUTATIVE ALGEBRAS WHICH ARE POLYNOMIAL ALGEBRAS

By R. F. Rinehart

1. **Introduction.** A polynomial $p(x)$ of non-zero degree with coefficients in an arbitrary field F gives rise to a linear algebra P, with a principal unit, over F. P may be viewed from two standpoints: (1) as the algebra generated by an element x whose minimum equation is $p(x) = 0$; (2) as the algebra of the residue classes modulo $p(x)$ of the ring of all polynomials with coefficients in F. From the first standpoint the elements $1, x, x^2, \cdots, x^{\alpha-1}$, where α is the degree of $p(x)$, constitute a basis for P; from the second standpoint the residue classes $[1]$, $[x]$, $[x^2], \cdots, [x^{\alpha-1}]$ constitute a basis. In either case P, considered as an abstract algebra, has the same properties. We shall call such an algebra *the polynomial algebra generated by* $p(x)$.

This paper had its origin in the speculation as to whether every commutative algebra[1] with a principal unit might be equivalent to a polynomial algebra.[2] The question is here answered in the negative, but it is found that the algebras which are thus completely characterized by polynomials constitute a wide class of commutative algebras. Under proper restrictions, depending on the nature of the ground field, it is shown that the equivalence of a commutative algebra with a principal unit to a polynomial algebra depends only on the structure of the radical. This structure is most conveniently described in terms of the écarts of certain nilpotent subalgebras, a concept which plays a prominent rôle in some recent researches of Scorza.

The results to follow are established under very loose hypotheses on the ground field F, namely, that F is separable, and that F, in case it is finite, has more elements than the commutative algebra in question has indecomposable Peirce components of a common order, for every order. When F is inseparable, the present analysis of the structure of the irreducible polynomial algebra, on which the treatment in §§3 and 4 is fundamentally based, fails. Moreover, as the examples of §5 show, the results of §§3 and 4 are not true for an inseparable ground field without imposition of further restrictions. The writer hopes to investigate the inseparable case later.

2. **Converses of the decomposition theorem for polynomial algebras.** Let P be the algebra generated by the polynomial $p(x)$ with coefficients in F, and let

Received April 20, 1938; presented to the American Mathematical Society, December 29, 1937.

[1] Throughout this paper the term algebra will be used to signify a linear associative algebra of finite order.

[2] Two algebras are said to be equivalent if a simple ring isomorphism exists between the elements of one algebra and the elements of the other.