INTERIOR SURFACE TRANSFORMATIONS

By G. T. Whyburn

In a previous paper[1] it was shown that when a 2-dimensional manifold A (with or without boundary curves) undergoes a light interior transformation $T(A) = B$, the resulting image B is likewise a 2-dimensional manifold. In this paper it will be shown that under these circumstances the Euler characteristics of the original and resulting manifolds are connected by a simple numerical relationship involving integers dependent only on A, B and the transformation T (see §2). Numerous examples and applications of this result follow in §§3 and 4. In the concluding section there is developed a method which effects the extension of this as well as other results concerning interior light transformations to the case of 2-dimensional pseudo-manifolds.

1. We consider a light interior transformation $T(A) = B$, where A (hence also[1] B) is a compact 2-dimensional manifold. Let α and β denote the boundaries (if any) of A and B, respectively. By a previous theorem[2] it follows that there exists an integer k such that the inverse of every point in B consists of k or fewer points. We define the least such integer k to be the *degree* of T. In other words, the degree k of T is the maximum multiplicity of T. Also from the theorem just cited it follows that there is only a finite number of points of $B - \beta$ whose inverse contains a point where T is not locally topological. Thus if Z denotes the set of all such points of $B - \beta$, it follows that on the set $A - T^{-1}(Z) - T^{-1}(\beta)$, T is locally topological; and since this set is connected, T must[3] be exactly k to 1 on this set. Thus k might also be defined as the multiplicity of T on the set $A - T^{-1}(\beta + Z)$.

Throughout this section the letters used above will retain their significance as there defined. Also $\chi(X)$ will stand for the Euler characteristic of the complex or surface X. We proceed to establish two lemmas.

Lemma 1. *If $N \subset B$ is a graph dividing B into only a finite number of components, then $T^{-1}(N)$ is a graph.*

Proof. Since all but a finite number of points of N are of order 2, $T^{-1}(N)$ has at most a finite number of end points. Since $A - T^{-1}(N)$ has only a finite number of components (each of which maps onto a component of $B - N$), it follows that $T^{-1}(N)$ contains only a finite number of simple closed curves.

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[1] See my paper in the American Journal of Mathematics, vol. 60(1938), pp. 477–490.

[2] Ibid., Theorem (5.2).

[3] See Eilenberg, Fundamenta Mathematicae, vol. 24(1935), p. 36.