INTEGRAL INEQUALITIES CONNECTED WITH DIFFERENTIAL OPERATORS

By I. Halperin and H. R. Pitt

1. **Introduction.** Suppose that the measurable functions $f(x)$, $f_r(x)$, $q_r(x)$, $q_{r,j}(x)$, $p_r(x)$, $c(x)$ are defined in the finite or infinite interval $\alpha < x < \alpha + a$, that $f_0(x), f_1(x), \cdots, f_{n-1}(x)$ are absolutely continuous in every closed subinterval and that almost everywhere[1] in $(\alpha, \alpha + a)$

$$\begin{aligned} f_0(x) &= q_0(x)f(x), \\ f_{r+1}(x) &= f_r'(x) + \sum_{j=0}^{r} q_{r,j}(x)f_j(x) + q_{r+1}(x)f(x) \qquad (r = 0, 1, \cdots, n-1), \end{aligned} \tag{1.1}$$

$$Tf(x) = \sum_{r=0}^{n} p_r(x)f_r(x) + c(x)f(x), \tag{1.2}$$

$$\begin{aligned} &|q_{r,j}(x)| \leqq M_1 && (j = 0, 1, \cdots, r; r = 0, 1, \cdots, n-1), \\ &|q_r(x)| \leqq M_1, \qquad |p_r(x)| \leqq M_2, && (r = 0, 1, \cdots, n), \\ &|c(x)| \leqq M_3, \end{aligned} \tag{1.3}$$

$$|q_0(x)| \geqq \epsilon > 0, \qquad |p_n(x)| \geqq \epsilon > 0. \tag{1.4}$$

We write

$$A_r = \left[\int_\alpha^{\alpha+a} |f_r(x)|^p \, dx\right]^{1/p}, \qquad B = \left[\int_\alpha^{\alpha+a} |Tf(x)|^p \, dx\right]^{1/p}, \tag{1.5}$$

where $p \geqq 1$.

Our object in this paper is to prove inequalities of the form

$$A_r \leqq F(A_0, A_n), \qquad A_r \leqq F(A_0, B), \tag{1.6}$$

and to use them to prove certain results about differential operators.

2. **Integral inequalities.**

(2.1) Theorem 1. *Suppose that* (1.1), (1.2), (1.3), *and* (1.4) *are satisfied and that* $\eta > 0$. *Then*

$$A_r \leqq \eta B + KA_0 \qquad (r = 0, 1, \cdots, n-1), \tag{2.1.1}$$

Received June 10, 1938.

[1] The words "almost everywhere" will be understood in similar statements that occur in what follows, although they will be frequently omitted.