BOOLEAN FUNCTIONS OF BOUNDED VARIATION

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Introduction. The properties of the relation

$$a \Delta b = ab' + a'b$$

suggest its use in framing a definition of a Boolean function of bounded variation similar to that for the real function of a real variable.[1] It is the purpose of this paper to frame such a definition and deduce the restrictions imposed on the function by it. These conditions in turn make possible a re-interpretation of many of the properties of the Δ-relation in terms of this restricted class of functions.

I. Functions of one variable

Definition 1. *The Boolean function $f(x)$ is said to be of bounded variation in the domain $(0, B)$ provided the sum*

$$\sum_{\substack{i=1 \\ j=1}}^{n} [f(\alpha_i) \Delta f(\alpha_j)] \tag{1}$$

is different from 1 *(the universal class) for all α_i, α_j subject to the conditions*

$$0 < \alpha_i < B, \tag{2}$$

$$\sum_{i=1}^{n} \alpha_i = B, \tag{3}$$

$$\alpha_i \alpha_j = 0 \qquad \text{for } i \neq j. \tag{4}$$

Definition 2. *If the sum* (1) *is null* $(= 0)$, *$f(x)$ is said to be an improper function of bounded variation in the domain $(0, B)$.*

Theorem I. *A necessary and sufficient condition that the function (in normal form)*

$$f(x) = ax + bx'$$

be a function of bounded variation in the domain $(0, B)$ is

$$(a \Delta b)B \neq 1. \tag{5}$$

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[1] For a detailed discussion of the properties of the Δ-relation, see Stone, *Postulates for Boolean algebras and generalized Boolean algebras*, American Journal of Mathematics, vol. 57(1935), pp. 703–732.