SIMPLE LIE ALGEBRAS OVER A FIELD OF CHARACTERISTIC ZERO

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The present paper gives a resumé and extension of the theory of simple Lie algebras over an arbitrary field Φ of characteristic 0 developed in recent papers by Landherr and by the author.[1] The extension consists in part in dropping the restriction of normality. Isomorphisms and automorphisms of simple but not necessarily normal simple Lie algebras are considered. We have also discussed the problem of cogredience of matrices arising in this connection and in the last sections have considered in detail the theory for a real closed field and sketched the theory also for $\mathfrak{p}$-adic fields. In the latter discussion we have confined ourselves to referring to results in the literature that are applicable here and have merely supplemented these with the theory of Hermitian and skew-Hermitian matrices having elements in a quaternion algebra.

1. **Preliminaries.** If $\mathfrak{A}$ is an associative algebra over a field Φ, it is readily seen that the elements of $\mathfrak{A}$ constitute a Lie algebra $\mathfrak{A}_l$ relative to the composition $[a, b] = ab - ba$. Evidently if $\mathfrak{A} \cong \mathfrak{B}$, $\mathfrak{A}_l \cong \mathfrak{B}_l$ and if $a \rightarrow b$ is an anti-isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, then $a \rightarrow -b$ is an isomorphism between $\mathfrak{A}_l$ and $\mathfrak{B}_l$. In particular if S is an automorphism (anti-automorphism) in $\mathfrak{A}$, then S $(-S\colon a \rightarrow -a^S)$ is an automorphism in $\mathfrak{A}_l$. It is clear that the elements left invariant by an automorphism in a Lie algebra form a subalgebra. Hence if S is an anti-automorphism in $\mathfrak{A}$, the set $\mathfrak{S}_S$ of S-skew elements b $(b^S = -b)$ form a Lie subalgebra of $\mathfrak{A}_l$.

The anti-automorphisms S and T of $\mathfrak{A}$ over Φ are *cogredient* if there exists an automorphism G of $\mathfrak{A}$ over Φ, such that $T = G^{-1}SG$. In this case G maps $\mathfrak{S}_S$ on $\mathfrak{S}_T$ and hence G is an isomorphism between these Lie algebras. If $S = T$, i.e., $SG = GS$, then G induces an automorphism in $\mathfrak{S}_S$. If G is inner, say $a^G = g^{-1}ag$, the condition $GS = SG$ is equivalent to $gg^S \in$ the centrum of $\mathfrak{A}$.

If $\mathfrak{L}$ is any (Lie) subalgebra of $\mathfrak{A}_l$, we define the *enveloping ring* $\mathfrak{R}$ of $\mathfrak{L}$ in $\mathfrak{A}$ to be the smallest subring of $\mathfrak{A}$ containing all the elements of $\mathfrak{L}$. $\mathfrak{R}$ is clearly the totality of elements of the form $\sum l_1 l_2 \cdots l_k$, where $l_i \in \mathfrak{L}$. Since $l\alpha \in \mathfrak{L}$ for any l in $\mathfrak{L}$ and any α in Φ, it is evident that $\mathfrak{R}$ is an algebra over Φ.

If $\mathfrak{L}$ is an arbitrary Lie algebra over Φ, it is well known that the correspondence between the element a in $\mathfrak{L}$ and the linear transformation $\mathbf{A}$ defined by $x\mathbf{A} \equiv [x, a]$ for x variable in $\mathfrak{L}$ is a representation, called the *adjoint representation,* of $\mathfrak{L}$:

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[1] Landherr [1], Jacobson [1] and [2]. Numbers in brackets refer to the bibliography at the end of the paper.