A LATTICE FORMULATION FOR TRANSCENDENCE DEGREES AND p-BASES

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1. **Introduction.** The transcendence degree of an extension of a field is the cardinal number of a maximal set of independent transcendents in the extension (Steinitz [11]);[1] in an Abelian group without elements of finite order the *rank* is the cardinal number of a maximal set of rationally independent group elements (Baer [1]). Both these cardinal numbers are invariants; the proofs of these two facts are similar, so that there should be an underlying theorem generalizing these proofs and stated in terms of the lattice of subfields or of subgroups, as the case may be.[2] This paper constructs, in §2, a type of lattice, called an "exchange" lattice, in which such a theorem can be proved (cf. §3). This lattice theorem includes also some investigations of Teichmüller ([12] and [13]) on fields of characteristic p; in particular, we establish the invariance of the cardinal number of a "relative p-basis" of an inseparable algebraic extension of such a field.

The crucial axiom for our lattices is an "exchange" axiom, related to the Steinitz exchange theorem. This axiom is equivalent to one of the axioms recently used by Menger [7] in investigating the algebra of affine geometry, and also to a certain covering property used by Birkhoff [2] in an analysis of the Jordan Theorem (cf. §4). This axiom can be viewed as a weakened form of the Dedekind or modular axiom for a lattice (Ore [9], or Birkhoff [4]). The Dedekind axiom itself could not apply to the lattices of fields with which we are concerned (see §5 and §6).

Unfortunately the exchange axiom is stated in terms of the "points" of the lattice, or alternately in terms of the covering relation. It thus applies only trivially to continuous geometries or to other infinite lattices having no points. In §7 we succeed in constructing two exchange axioms which in the presence of the other axioms are equivalent to the original exchange axiom, but which themselves do not involve points or coverings. These new axioms yield most of the usual properties of the dimension function in a finite lattice. Their title to be considered as substitutes for the modular law rests chiefly on their versatility: each of the new exchange axioms is seen in §8 to be equivalent to a natural assertion about the possibility of specified types of transpositions in any given chain of the lattice.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] This remark is due to R. Baer (in conversation).