ORTHOGONAL POLYNOMIALS IN THREE VARIABLES

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1. **Introduction.** The theory of orthogonal polynomials in two real variables naturally carries over to a considerable extent automatically to the case of three or more variables. For an adequate survey of the facts, nevertheless, it is necessary in some particulars to take explicit account of the greater complexity introduced by the increased number of dimensions. The Laplace series, for example, which can be regarded as an expansion in series of orthogonal polynomials on a sphere in space, is neither formally nor analytically a trivial extension of the Fourier series, which corresponds similarly to a circle in the plane.

This paper is introductory to the study of a class of developments among which the Laplace series is included as a special case, the primary aim being to clarify some of the new considerations that arise in making the transition from two variables[1] to a larger number. One question in particular relates to the determination of the number of polynomials of the n-th degree in the orthogonal system, a point which naturally has a dominant influence on the form of the resulting series developments. The discussion will be mainly for three variables, partly because the general outline of the extension to more than three will be sufficiently apparent, and partly, on the other hand, because a complete elucidation even of the three-dimensional case would be more than can be attempted here.

The range of integration with relation to which the property of orthogonality is defined may be a point set of more or less arbitrary character. It will be sufficient for purposes of illustration to take it as of finite extent, and to think of it as a region of space, a surface, or a curve. There will be occasion to distinguish between algebraic and non-algebraic surfaces, and in the case of curves to distinguish between algebraic curves, non-algebraic curves on an algebraic surface, and curves which do not lie on any algebraic surface. The curves and surfaces may or may not be closed. Algebraic curves and surfaces need not be complete algebraic loci; the surface considered may be the surface of a polyhedron, a hemispherical surface, or the complete surface of a solid hemisphere; the curve may be a skew polygon.

Occasion will be taken to refer to one matter, the uniqueness of the weight function for a given orthogonal system, which has not been discussed before even for the two-dimensional problem.

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[1] See D. Jackson, *Formal properties of orthogonal polynomials in two variables*, this Journal, vol. 2(1936), pp. 423–434; *Orthogonal polynomials on a plane curve*, this Journal, vol. 3(1937), pp. 228–236.