THE JUMP OF A FUNCTION DETERMINED BY ITS FOURIER COEFFICIENTS

BY OTTO SZÁSZ

1. Let $f(x)$ be integrable L in the interval $(-\pi, \pi)$ and have period 2π, and let its Fourier series be

$$f(x) \sim \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x),$$

where

$$2c_\nu = a_\nu - ib_\nu = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) e^{-i\nu t} \, dt \qquad (\nu = 0, 1, 2, \cdots). \tag{1}$$

We shall also use

$$\begin{aligned} s_n(x) &= \frac{a_0}{2} + \sum_{1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x) \equiv \sum_{0}^{n} A_\nu(x), \\ \bar{s}_n(x) &= \sum_{1}^{n} (b_\nu \cos \nu x - a_\nu \sin \nu x) \equiv \sum_{1}^{n} \bar{A}_\nu(x). \end{aligned} \tag{2}$$

$\bar{s}_n(x)$ is the trigonometric polynomial conjugate to $s_n(x)$. It is well known that the arithmetic means

$$\begin{aligned} \sigma_n(x) &= \frac{s_0 + s_1 + \cdots + s_{n-1}}{n} = \frac{1}{n} \sum_{0}^{n} (n - \nu) A_\nu \\ &= \frac{1}{2n\pi} \int_{-\pi}^{\pi} f(t) \left(\frac{\sin \frac{1}{2} n(t - x)}{\sin \frac{1}{2}(t - x)} \right)^2 dt \end{aligned} \tag{3}$$

converge to $\frac{1}{2}\{f(x + 0) + f(x - 0)\}$, whenever this expression exists.[1] We are concerned here with the determination of the jump: $D(x) = f(x + 0) - f(x - 0)$.

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[1] This is Fejér's classical theorem; it was generalized by Lebesgue, who proved $\sigma_n(x) \to f(x)$ whenever

$$\frac{1}{h} \int_0^h | f(x + t) + f(x - t) - 2f(x) | \, dt \to 0 \qquad \text{as } h \to 0.$$

An extension of this result is [3, §5; 6; 4]:

$$\int_0^h | \varphi(t) | \, dt = O(h) \quad \text{and} \quad \int_0^h \varphi(t) \, dt = o(h),$$

where $\varphi(t) = f(x + t) + f(x - t) - 2f(x)$, imply $\sigma_n(x) \to f(x)$.