DEGREE OF APPROXIMATION BY POLYNOMIALS IN z AND $1/z$

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1. **Introduction.** A polynomial of degree n in z and $1/z$ is a function of the form

$$r_n(z) = a_{-n}z^{-n} + a_{-n+1}z^{-n+1} + \cdots + a_{-1}z^{-1} + a_0 + \cdots + a_nz^n; \tag{1.10}$$

we do not assume a_{-n} or a_n different from zero. Riesz[1] has shown that $| r_n(z) | \leqq M$ on C: $| z | = 1$ implies $| r_n'(z) | \leqq Mn$, $| z | = 1$. In this paper we extend this result to various types of Jordan curves (see §2) for a generalized derivative (see §3) of an arbitrary positive order α. In fact, we prove that *if C is a Jordan curve containing the origin in its interior, then $| r_n(z) | \leqq M$, for z on C, implies*[2] $| r_n^{\alpha}(z) | \leqq MK(\alpha, C)n^{\alpha u}$, $\alpha > 0$, $1 \leqq u \leqq 2$, *where K is a constant depending only on α and C, and u is a constant depending only on C.*

Also let $f(z)$ be defined on C and suppose $|f(z) - r_n(z)| \leqq \epsilon_n$, z on C $(n = 1, 2, \cdots)$. If $f(z)$ is continuous on C, there exists[3] for each n a polynomial $r_n(z)$ such that ϵ_n approaches zero as n becomes infinite. Here we study the relation between ϵ_n and the continuity properties of $f(z)$ on C. For an analytic Jordan curve C (see §4) the method consists in mapping the interior of C conformally on $| w | < 1$ and applying results on trigonometric approximation due to de la Vallée Poussin[4] and Jackson.[5] We prove, for example, that, *for C an analytic Jordan curve, the existence of $r_n(z)$ $(n = 1, 2, \cdots)$ such that $|f(z) - r_n(z)| \leqq Mn^{-\alpha}$, z on C, $0 < \alpha < 1$, α and M independent of n and z, implies that $f(z)$ satisfies a Lipschitz condition*[6] *of order α on C, and, conversely, $f(z)$ satisfying a Lipschitz condition of order α on C implies the existence of $r_n(z)$ such that $|f(z) - r_n(z)| \leqq Mn^{-\alpha}$, z on C.* For $f(z)$ the boundary function of a function

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[1] M. Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23(1914), pp. 354–368.

[2] $f^{\alpha}(z)$ denotes the generalized derivative of order α of $f(z)$.

[3] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 20, 1935; see p. 38.

[4] Ch.-J. de la Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle*, Paris, 1919.

[5] Dunham Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. 11, 1930.

[6] The function $f(z)$ satisfies a Lipschitz condition of order α on C if for z_1 and z_2 arbitrary points on C we have $| f(z_1) - f(z_2) | \leqq L | z_1 - z_2 |^{\alpha}$, where L is a constant independent of z_1 and z_2.