AN AFFINE INVARIANT OF CONVEX REGIONS

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1. **Introduction.** Let R denote a closed and bounded three-dimensional convex region. Let D be the greatest diameter (Durchmesser), and let Δ be the smallest diameter (Dicke) of R.[1] We speak of convex regions as belonging to different *classes*, a single class consisting of those regions which can be derived from one another by affine transformations.

The problem treated in this paper is that of finding the maximum for all classes C of the minimum values of D/Δ for all regions l belonging to C; i.e., to find $\max_{C} \min_{l \subset C} D/\Delta$. This problem has been solved originally by F. Behrend[2] and subsequently by F. John[3] for the two-dimensional case. They find $\max_{C} \min_{l \subset C} D/\Delta = \sqrt{2}$, and the class of parallelograms is the only class which attains this maximum. In the three-dimensional case, we find that the constant is $\sqrt{3}$, and that, of convex regions having a center, there are *two* classes which attain this maximum, namely, the parallelepipeds and the octahedrons.[4] If we measure *similarity of a region to a sphere* by the value D/Δ, our result states how similar to a sphere a convex region can be made by an affine transformation.

2. **A necessary condition for minimal regions.** Let us call a region l_1 of the class C for which D/Δ is a minimum a *minimal region* of the class. We first derive a necessary condition that the convex region l_1 shall be a minimal region.

We suppose that the convex region R has a *center*, i.e., R is symmetrical with respect to some point. We will consider later convex regions without center, a modification which presents no difficulty. In the present case, $D/\Delta = 1$ only for spheres, and we note that $\min_{C} \min_{l \subset C} D/\Delta = 1$ is attained only by the class of ellipsoids. This is a trivial result.

Take the center of R as the origin of a rectangular system of coördinates. The distance from the center to any point on the boundary of R we shall call a *radius*, r, of R. Then, $\frac{1}{2}D = \max r$, and $\frac{1}{2}\Delta = \min r$. We apply to the

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[1] A diameter of R is the distance apart of two parallel planes of support. For definitions, see Bonnesen-Fenchel, *Theorie der konvexen Körper*, Ergebnisse der Mathematik, vol. 3, No. 1, Berlin, 1934, p. 37 ff.

[2] F. Behrend, *Über einige Affininvarianten konvexer Bereiche*, Math. Annalen, vol. 113, pp. 713–747.

[3] F. John, *Moments of inertia of convex regions*, this Journal, vol. 2(1936), pp. 447–452.

[4] A proof that the constant in question is $\leqq \frac{1}{3}\sqrt{30}$ is given by John, loc. cit. The methods used in the present paper are more closely related to those of Behrend.