CHARACTERIZATION OF THE CONFORMAL GROUP AND THE EQUI-LONG GROUP BY HORN ANGLES

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We begin by giving certain preliminary definitions. A *horn angle* consists of two curves which pass through a common point in a common direction. In this paper we consider only horn angles of first order contact, that is, the curves of the horn angle have different curvatures at the common point. By a *contact transformation* we mean a lineal element transformation by which every curve (union) corresponds to a curve. Thus it follows that every contact transformation converts every horn angle into a horn angle. In our previous work[1] we have defined a *natural conformal measure* M_{12} and a distinct *natural equi-long measure* μ_{12} of a horn angle. In this paper we shall determine all contact transformations that preserve M_{12} or μ_{12}. Our results are that the group for which M_{12} is invariant is the direct conformal group; and, dually, the group for which μ_{12} is invariant is the direct equi-long group.

Separate proofs are required for these two results. For, although the two theories are analogous (or roughly dual), they are not connected by any known automatic principle of duality or transference principle. In fact we will notice in the course of our work that some of our preliminary theorems lead to results which are not strictly dual.

We shall consider the geometric properties of certain three-parameter families of curves designated as L-families and λ-families. They may be regarded as generalizations of dynamical trajectories. One of our principal results is that the group of transformations which convert every L-family of curves into an L-family of curves is the conformal group. On the other hand, the group of transformations which convert every λ-family of curves into a λ-family of curves is a group of line transformations which convert parallel lines into parallel lines. This group is much larger than the equi-long group, so that our two results are not dual. The dynamical type group is projective.

Finally we shall prove that there is no contact transformation which changes every conformal measure M_{12} into an equal equi-long measure μ_{12}. This is an

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[1] Proceedings of the International Congress of Mathematicians, Cambridge, No. 2 (1912), p. 81.

Proceedings of the National Academy of Sciences (with George Comenetz), vol. 22 (1936), p. 303.

Science, vol. 85 (1937), p. 480.

Proceedings of the National Academy of Sciences, vol. 23 (1937), p. 337.

Annals of Mathematics, October, 1937.

Transactions of the American Mathematical Society (1938).