QUASI-UNITARY MATRICES

By John Williamson

Introduction. Let I_m be the n-rowed square matrix

$$\begin{pmatrix} E_m & 0 \\ 0 & -E_{n-m} \end{pmatrix},$$

where E_j is the unit matrix of order j. Then I_m is the normal form of a non-singular Hermitian matrix of index m under a non-singular conjunctive transformation. A matrix A, whose elements are complex numbers, which satisfies

$$AI_mA^* = I_m, \tag{1}$$

where $A^* = \bar{A}'$ is the conjugate transposed of A, will be called a *quasi-unitary matrix*. In particular, if $m = n$ or 0, A is a unitary matrix. A matrix, A, which satisfies (1), is a conjunctive automorph of the Hermitian matrix I_m. The conjunctive automorphs of a non-singular Hermitian matrix have been studied by Loewy.[1] He has shown how the nature of the elementary divisors of $A - \lambda E$ is restricted by the index m of the matrix I_m. In the following paper we derive normal forms for quasi-unitary matrices under quasi-unitary transformations, and in doing so are inevitably led to Loewy's results. (See, for example, the remark following Theorem 2.) We also determine necessary and sufficient conditions for the similarity of two quasi-unitary matrices under a quasi-unitary transformation. In particular it is shown that two quasi-unitary matrices which are similar are not necessarily similar under a quasi-unitary transformation. In §2 the similar problem for real quasi-orthogonal matrices is considered, and in §4 an interesting property of the elementary divisors of a pencil, whose base is I_m and a canonical quasi-unitary matrix, is deduced.

As many of the proofs are in essence the same, subject to obvious modifications, as those in a previous paper,[2] for the sake of brevity they will be omitted.

Received May 4, 1937; in revised form September 21, 1937.

[1] Alfred Loewy, *Allgemeine bilineare Formen konjugirt imaginären Variabeln*, Abhandlungen der Kaiserlichen Leopoldinisch-Carolinischen Deutschen Akademie der Naturforscher, vol. 71 (1898), pp. 377–446; Mathematische Annalen, vol. 50, pp. 557–576. The second of these papers gives a short account of the results proved in the first. The term quasi-unitary was first used by Harold Hilton, *Properties of certain homogeneous linear substitutions*, Annals of Mathematics, (2), vol. 15 (1913), pp. 195–201.

[2] John Williamson, *On the normal forms of linear canonical transformations in dynamics*, American Journal of Mathematics, vol. 59 (1937), pp. 599–617. This paper will be referred to as W.