TRANSFORMATIONS ON SEQUENCE SPACES

By L. W. Cohen and Nelson Dunford

Hardy and Littlewood,[1] Littlewood[2] and others have given certain necessary and other sufficient conditions on the matrix a_{ij} in order that the bilinear form $\Sigma\Sigma a_{ij}x_i x_j$ be bounded for $\Sigma \mid x_i \mid^p \leqq 1$, $\Sigma \mid y_i \mid^q \leqq 1$. So far as we know no conditions on the matrix a_{ij} alone have been given which are necessary as well as sufficient for the boundedness of the corresponding bilinear form. In this paper we consider among other questions the more precise problem of determining the norm of the linear transformation $y = Tx$ on l_p to l_q in terms of the elements a_{ij} of the matrix representing this transformation. We have been successful in the special cases where T is on l_1 to l_p or c_0 and, less trivially, on l_p or c_0 to l_1, if $a_{ij} \geqq 0$. Conditions for the absolute convergence of the determinant of $(\delta_{ij} + a_{ij})$ representing $I + T$ as well as properties of the matrix of minors are also obtained. These last conditions together with necessary and sufficient conditions for compactness have been given for a Banach space with a denumerable basis φ_n. In such a space each element x is uniquely representable as

$$x = \sum_{n=1}^{\infty} x_n \varphi_n, \qquad x_n = T_n^{\Phi} x,$$

where T_n^{Φ} is a linear functional on the space Φ with $\mid T_n^{\Phi} \mid \leqq M_{\Phi}$. In such a space the convergence of x^m to x implies the uniform convergence of x_n^m to x_n.

In view of the well-known theorems on uniform boundedness of sequences of linear operations and the known conditions for weak convergence in many Banach spaces, it is comparatively trivial to give the form and norm of the general linear operation with the range in c, $m = l_\infty$, C, M (bounded functions), etc. Consequently these cases have been omitted from the discussion of such questions.

Theorem 1. *If Φ and Ψ are Banach spaces with denumerable bases and $Tx = y$ is a linear transformation of Φ into Ψ, the transformation is represented by*

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j, \tag{1}$$

Received June 2, 1937; presented to the American Mathematical Society, December, 1936, under the titles: L. W. Cohen, *Transformations on spaces with denumerable basis*; Nelson Dunford, *Linear transformations of sequences.*

[1] G. H. Hardy and J. E. Littlewood, *Bilinear forms bounded in space* [p, q], Quarterly Journal of Math., (Oxford), vol. 5 (1934), pp. 241–254.

[2] J. E. Littlewood, *On bounded bilinear forms in an infinite number of variables*, Quarterly Journal of Math., (Oxford), vol. 1 (1930), pp. 164–174.