REMARKS ON THE PROBLEM OF PLATEAU

By E. F. Beckenbach

1. **Introduction.** We shall consider the problem of Plateau in the following form.

Problem of Plateau. *Given a Jordan curve* Γ *in xyz-space, determine functions* $x(u, v)$, $y(u, v)$, $z(u, v)$ *which are continuous for* $u^2 + v^2 \leqq 1$, *are harmonic and satisfy* $E = G$, $F = 0$, *where*

$$E = x_u^2 + y_u^2 + z_u^2 , \quad F = x_u x_v + y_u y_v + z_u z_v , \quad G = x_v^2 + y_v^2 + z_v^2 ,$$

for $u^2 + v^2 < 1$, *and map* $u^2 + v^2 = 1$ *in a topological way on* Γ.

Any set of functions satisfying the above conditions are coördinate functions of a minimal surface bounded by Γ and given in isothermic representation.

The following theorems have been proved.

Theorem 1. *If* Γ *bounds some surface, of the type of the circular disc, with a finite area, then the problem of Plateau is solvable for* Γ.

Theorem 2. *The problem of Plateau is solvable for an arbitrary Jordan curve* Γ.

Theorem 1 has been proved separately and at about the same time by J. Douglas and T. Radó.[1] Subsequent proofs have been given by E. J. McShane[2] and R. Courant.[3] Theorem 2 has been proved by J. Douglas (loc. cit.), and later, by means of a different method but the same lemmas, by T. Radó.[4] In what follows, we consider alternative proofs of this latter theorem.

In proving Theorem 2, Douglas, assuming Theorem 1, first uses a limiting process to establish the existence of mapping functions. He then completes the proof by using the following two lemmas to show that the functions thus obtained map $u^2 + v^2 = 1$ topologically on Γ.

Lemma 1. *Let* $x(u, v)$, $y(u, v)$, $z(u, v)$ *be harmonic and satisfy* $E = G$, $F = 0$ *for* $u^2 + v^2 < 1$. *Suppose* $x(u, v)$, $y(u, v)$, $z(u, v)$ *remain continuous on an arc* σ *of* $u^2 + v^2 = 1$, *and* $x(u, v) = const. = x_0$, $y(u, v) = const. = y_0$, $z(u, v) = const. = z_0$ *on* σ. *Then* $x(u, v) \equiv x_0$, $y(u, v) \equiv y_0$, $z(u, v) \equiv z_0$.

Douglas' Lemma. *Let the integrable functions* $\xi(\varphi)$, $\eta(\varphi)$, $\zeta(\varphi)$, *substituted*

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[1] Their results are summed up in the following papers: J. Douglas, *Solution of the problem of Plateau*, Transactions of the American Mathematical Society, vol. 33 (1931), pp. 263–321; T. Radó, *The problem of the least area and the problem of Plateau*, Mathematische Zeitschrift, vol. 32 (1930), pp. 763–796.

[2] E. J. McShane, *Parametrization of saddle surfaces, with application to the problem of Plateau*, Transactions of the American Mathematical Society, vol. 35 (1933), pp. 716–733.

[3] R. Courant, *On the problem of Plateau*, Proceedings of the National Academy of Sciences, U. S. A., vol. 22 (1936), pp. 367–372.

[4] *An iterative process in the problem of Plateau*, Transactions of the American Mathematical Society, vol. 35 (1933), pp. 869–887.