ASYMPTOTIC EXPRESSIONS FOR THE ZEROS OF GENERALIZED LAGUERRE POLYNOMIALS AND WEBER FUNCTIONS

By Vivian Eberle Spencer

Introduction. It is the purpose of this paper to apply the close relationship between Hermite and Laguerre polynomials to find asymptotic expressions for the zeros $\{_L x_{in}\}$ of the generalized Laguerre polynomial $L_n(x, \alpha)$. The results obtained for the largest zero $_L x_{nn}$ are, we believe, new; for the other zeros the expressions are essentially equivalent to those obtained by Winston;[1] the method of procedure, however, in every case is new and fruitful. Hermite functions $h(x, n)$ are the special case of Weber's parabolic cylinder functions $w(x, n)$ obtained when the boundary conditions $w(\pm\infty, n) = O(x^n e^{-\frac{1}{4}x^2})$ are imposed. The argument is simplified and the order of some of the results improved by the introduction of the latter functions for non-integral n. Hence we are led to a discussion of the zeros $\{_w x_{in}\}$ of $w(x, n)$. By an elementary application of Sturm's theory Milne's[2] properties of the zeros $\{_w x_{in}\}$ of the standard solution are obtained, and similar properties for the zeros of the solution converging to zero as x approaches minus infinity are developed. The application of the known asymptotic expressions for the zeros of Hermite polynomials is shown to give directly bounds and asymptotic expressions for the zeros of Weber functions for n an arbitrary positive number. A sequence of Weber functions $\omega(x, \alpha, n)$ is associated with every sequence of Laguerre functions $l(x, \alpha, n)$, and definite separation and asymptotic relations are obtained between the zeros of $\omega(x, \alpha, n)$ and $l(x, \alpha, n)$. As a consequence of these relations asymptotic expressions for all zeros of $L_n(x, \alpha)$, for any $\alpha > 0$, are obtainable immediately from any asymptotic expression for the zeros of the Hermite polynomial $H_n(x)$, or for the zeros of the Laguerre polynomial proper $L_n(x, 1)$. Neumann's[3] bounds for the zeros of $L_n(x, 1)$ and Zernike's[4] asymptotic expression for the largest zero of $H_n(x)$ are then applied to $L_n(x, \alpha)$.

1. **An asymptotic expression for $_L x_{nn}$ for $\frac{1}{2} \leqq \alpha \leqq \frac{3}{2}$.** We consider the Hermite and Laguerre polynomials satisfying respectively the differential equations

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0, \tag{1}$$

$$xL_n''(x, \alpha) + (\alpha - x)L_n'(x, \alpha) + nL_n(x, \alpha) = 0 \qquad (\alpha > 0). \tag{2}$$

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[1] C. Winston, *On mechanical quadratures formulae involving the classical orthogonal polynomials*, Annals of Math., vol. 35 (1934), pp. 658–677.

[2] A. Milne, *On the roots of the confluent hypergeometric functions*, Proc. Edinburgh Math. Soc., vol. 33 (1915), pp. 48–64.

[3] E. R. Neumann, *Beiträge zur Kenntnis der Laguerreschen Polynome*, Jahresbericht der Deutschen Math.-Vereinigung, vol. 30 (1921), pp. 15–35.

[4] F. Zernike, *Eine asymptotische Entwicklung für die grösste Nullstelle der Hermiteschen Polynome*, Amsterdam Academy, Proc. of Sec. Sc., vol. 34 (1931), pp. 673–680.