## ASYMPTOTIC RELATIONS FOR DERIVATIVES

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1. **Introduction.** A well known theorem of Hardy and Littlewood states, in a form in which it is often quoted, that if $f(x)$ is of class $C^2$ on $(0, \infty)$, and if, as $x \to \infty$, $f(x) = o(1)$ and $f''(x) = O(x^{-2})$, then $f'(x) = o(x^{-1})$. It is a special case of a theorem in which the powers of $x$ in the order relations are replaced by more general functions; and this in turn can be used to establish an extended theorem where from the order of $f(x)$ and of $f^{(n)}(x)$ $(n \geqq 2)$ one deduces the orders of the intermediate derivatives.[1] Now, if we think of the hypothesis on $f''(x)$ in the original theorem as "$x^2 f''(x) = O(1)$", it is a hypothesis on the order of the function resulting from applying a certain linear differential operator to $f(x)$. The principal result of this note is the corresponding theorem when the operator $x^2 \dfrac{d^2}{dx^2}$ is replaced by a certain more general, $n$-th order, linear operator, $L$; from the order of $L[f(x)]$ and of $f(x)$, the order of

$$f^{(k)}(x) \qquad (k = 1, 2, \cdots, n-1)$$

can be deduced. This result, and a preliminary theorem, overlap the results of Hardy and Littlewood, but neither include them nor are included by them. The full statement of our main theorem is somewhat complex; to illustrate it as simply as possible, a special case, sufficient for many applications, will be stated here.

*Let*

$$L[f(x)] = \sum_{i=0}^{n} A_i x^i f^{(i)}(x), \tag{1.1}$$

*where the $A_i$ are constants, $A_n \neq 0$. Let $f(x)$ be of class $C^n$ on $(0, \infty)$, and suppose that as $x \to \infty$, $L[f(x)] < O(1)$. If $f(x) = O(1)$, then $f^{(k)}(x) = O(x^{-k})$; if $f(x) = o(1)$, then $f^{(k)}(x) = o(x^{-k})$ $(k = 1, 2, \cdots, n-1)$.*

Examples of operators $L[f(x)]$ which have the form (1.1) are $x^n f^{(n)}(x)$; the operator

$$L_{k,x}[f(x)] = \frac{(-x)^{k-1}}{k!\,(k-2)!} \frac{d^{2k-1}}{dx^{2k-1}} [x^k f(x)] \qquad (k \geqq 2)$$

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[1] G. H. Hardy and J. E. Littlewood, *Contributions to the arithmetic theory of series*, Proceedings of the London Mathematical Society, (2), vol. 11 (1913), pp. 411–478; 417 ff.

Reference should also be made to E. Landau, *Über einen Satz von Herrn Esclangon*, Mathematische Annalen, vol. 102 (1929–30), pp. 177–188. Landau considers more general differential operators than we do, but his results are less general in other respects.