# THE INVERSION PROBLEM OF MÖBIUS

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1. **Introduction.** The present paper represents an attempt to give a rigorous treatment of certain inversion problems which have their origin in a little-known paper by A. F. Möbius.[1]

As a typical, though not the oldest, example of these inversion problems we might take the linear functional equation with constant coefficients

$$\sum_{n=1}^{\infty} a_n f(nz) = g(z), \tag{1.1}$$

a formal solution of which has the form

$$\sum_{n=1}^{\infty} b_n g(nz) = f(z). \tag{1.2}$$

These problems all lead to the same infinite system of bilinear equations

$$a_1 b_1 = 1, \qquad \sum_{d \mid n} a_d b_{n/d} = 0, \quad n > 1, \tag{1.3}$$

for which the *algorithm of Möbius* seems a fitting name.

This algorithm is perhaps best known from the problem of finding the reciprocal of an ordinary Dirichlet series, i.e., a solution of the problem

$$\sum_{n=1}^{\infty} a_n n^{-s} \sum_{n=1}^{\infty} b_n n^{-s} = 1. \tag{1.4}$$

We shall see that the properties of these series are fundamental in all these inversion problems.

This observation suggests that there is a class of inversion problems associated with the problem of expressing the reciprocal of a general Dirichlet series or, still more generally, of a Laplace-Stieltjes integral as a function of the same class. In general the reciprocal is not so expressible, but whenever it is, certain functional equations of the type

$$\int_1^{\infty} f(uz)\, dA(u) = g(z) \tag{1.5}$$

have solutions of the form

$$\int_1^{\infty} g(uz)\, dB(u) = f(z), \tag{1.6}$$

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[1] *Ueber eine besondere Art von Umkehrung der Reihen*, Journal f. Math., vol. 9 (1832), pp. 105–123; *Gesammelte Werke*, vol. IV, 1887, pp. 589–612.