A NOTE ON NON-ASSOCIATIVE ALGEBRAS

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It is the purpose of this note to obtain relations between an arbitrary algebra $\mathfrak{R}$ (not necessarily associative) and an algebra $\mathfrak{A}$ (necessarily associative) of linear transformations determined by $\mathfrak{R}$. If $\mathfrak{A}$ is simple, the centrum $\mathfrak{C}$ of $\mathfrak{A}$ is an algebraic field and $\mathfrak{R}$ may be regarded as an algebra over $\mathfrak{C}$. When this is done $\mathfrak{R}$ becomes a *normal simple algebra*, i.e., remains simple when this field is extended to its algebraic closure. A field having this property for algebras of characteristic 0 has been defined previously but less directly by Landherr.[1] Some of our results have been announced for Lie algebras of characteristic 0 by Albert.[2]

1. Let $\mathfrak{R}$ be an arbitrary algebra (not necessarily associative or commutative) with a finite basis over a commutative field Φ; $\mathfrak{R}$ is a finite dimensional vector space over Φ in which there is defined a composition xy of pairs of elements x, y such that

$$(1) \qquad (x + y)z = xz + yz, \qquad z(x + y) = zx + zy,$$

$$(2) \qquad (xy)\alpha = x(y\alpha) = (x\alpha)y, \qquad \alpha \in \Phi.$$

The mapping $x \to xa \equiv xA_r$ of $\mathfrak{R}$ on itself will be called the *right multiplication* determined by a. Equations (1) and (2) show that A_r is a linear transformation in the vector space $\mathfrak{R}$. Similarly we define the *left multiplication* determined by a as $x \to ax \equiv xA_l$. Let $\mathfrak{A}$ be the enveloping algebra of the left and right multiplications of $\mathfrak{R}$, i.e., the smallest algebra of linear transformations in $\mathfrak{R}$ containing all the left and right multiplications. The elements of $\mathfrak{A}$ are sums of terms of the type $A_{1i_1} \cdots A_{si_s}$ ($i_\alpha = r$ or l) where A_{ji_j} is a multiplication determined by a_j. We shall therefore denote an arbitrary element of $\mathfrak{A}$ by $\Sigma A_{1i_1} \cdots A_{si_s}$ (not summed on i_α!). Thus $\mathfrak{A}$ may also be defined as the smallest ring of linear transformations containing all the multiplications.

If $a_1, \cdots, a_n$ is a basis for $\mathfrak{R}$ over Φ and A is a linear transformation in this vector space, then A is completely determined by the matrix (α_{ij}) such that $a_jA = \Sigma a_i\alpha_{ij}$. The correspondence between A and the matrix (α_{ij}) determines, as is well known, a reciprocal isomorphism between the ring of all linear transformations in $\mathfrak{R}$ over Φ and the matrix ring Φ_n of all n-rowed square matrices

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[1] W. Landherr, *Über einfache Liesche Ringe*, Hamb. Abhandlungen, vol. 11 (1935), pp. 41–64.

[2] Bull. Am. Math. Soc., vol. 41 (1935), p. 344.