SUMS OF VALUES OF A POLYNOMIAL MULTIPLIED BY CONSTANTS

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1. **Introduction.** We seek conditions on the integer s, on the sets of positive integers $(a_1, \cdots, a_s)$ and on the coefficients of a polynomial $P(x)$ for which the Diophantine equation

$$n = \sum_{\nu=1}^{s} a_\nu P(h_\nu) \tag{1}$$

is solvable in integers $h_\nu \geqq 0$ for every integer n sufficiently large. By n sufficiently large we mean that n is greater than an existing constant b_1 which depends only on $s, a_1, \cdots, a_s$ and on the degree k and the coefficients of the polynomial $P(x)$. We consider two cases of the polynomial $P(x)$:

$$P(x) = a(x^3 - x)/6 + b(x^2 - x)/2 + cx + d, \tag{2}$$

where $a > 0$ and a, b, c and d are integers;[1]

$$\begin{aligned} P(x) = {} & ax(x + 1)(x + 2)(x + 3)/24 + bx(x + 1)(x + 2)/6 \\ & + cx(x + 1)/2 + dx + e, \end{aligned} \tag{3}$$

where $a > 0$ and a, b, c, d and e are integers.[2]

For $a_1 = \cdots = a_s = 1$, the problem is the classical Waring problem for third and fourth degree polynomials. If $P(x)$ in (2) is such that $a \not\equiv 4c \pmod 8$ James has shown that every sufficiently large integer n is a sum of nine values of $P(x)$.[3] We show that for $s = 9$ and for integral constants $(a_1, \cdots, a_s)$ satisfying certain congruential conditions given later, every sufficiently large integer can be expressed in the form (1). For $P(x)$ as in (3), Miss Humphreys has given conditions which are sufficient to prove that every sufficiently large integer n is expressible as the sum of 21 values of $P(x)$. Under certain further assumptions on $P(x)$ and on the sets of positive integers $(a_1, \cdots, a_s)$ we shall

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[1] A cubic polynomial in x is an integer for all integers $x \geqq 0$ if and only if it is of the form (2)—R. D. James, *The representation of integers as sums of values of cubic polynomials*, American Journal of Mathematics, vol. 56 (1934), pp. 303–315. See also D. H. Hilbert, *Über die Theorie der algebraischen Formen*, Mathematische Annalen, vol. 36 (1890), pp. 511–512.

[2] A fourth degree polynomial in x is an integer for all integers $x \geqq 0$ if and only if it is of the form (3). See M. G. Humphreys, *On the Waring problem with polynomial summands*, this Journal, vol. 1 (1935), pp. 361–375. The proof is accomplished by a slight modification of that of Hilbert in the paper previously cited.

[3] James, loc. cit.