A STRUCTURAL CHARACTERIZATION OF PLANAR COMBINATORIAL GRAPHS

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1. **Introduction.** There are several known necessary and sufficient conditions that a combinatorial graph be planar.[1] This paper aims to establish another such condition which has a more intrinsic character, in that it is obtained directly from an analysis of the structure of the graph. More explicitly, the new condition depends on a unique decomposition of the graph into certain maximal triply connected subgraphs. This decomposition can be viewed on its own merits as a generalization of the Whyburn cyclic element theory.

The first combinatorial criterion for a planar graph is due to Kuratowski,[2] who showed that a graph is planar if and only if it contains no subgraph homeomorphic to one of the two following graphs: the graph composed of five vertices and ten edges, in which each pair of vertices is joined by an edge; the graph composed of six vertices, arranged in two sets of three vertices each, and of nine edges, such that each vertex of the first set is joined to each vertex of the second set by an edge. Subsequently, Whitney defined combinatorially the relation between a graph and its planar dual and showed that a graph is planar if and only if it has a planar dual.[3] A third condition states that a combinatorial graph is planar if and only if it contains a complete independent set of circuits, modulo 2, such that no edge appears in more than two of these circuits.[4]

The application of any of these theorems to a particular case has a haphazard character because one must investigate any possible smallest non-planar subgraph or any possible dual or any possible complete set of circuits. We seek an intrinsic condition;[5] that is, a condition expressible in terms of configurations which are associated in a unique manner with a given graph. An example of such a condition is the result of Whitney, that any graph G can be broken up uniquely into non-separable components[6] and that the graph is planar if and only if each of its non-separable components is planar.[7]

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[1] For definitions of terms see §2.

[2] K. Kuratowski, *Sur le problème des courbes gauches en topologie*, Fundamenta Mathematicae, vol. 15 (1930), pp. 271–283.

[3] H. Whitney, *Non-separable and planar graphs*, Transactions of the American Mathematical Society, vol. 34 (1932), pp. 339–362.

[4] S. Mac Lane, *A combinatorial condition for planar graphs*, Fundamenta Mathematicae, vol. 28 (1937), pp. 22–32.

[5] It is to be hoped that such a condition may throw light on the question of when a graph is mappable on the torus.

[6] These components of G are precisely the true cyclic elements of G. See G. T. Whyburn, *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), pp. 167–194. For the combinatorial decomposition into components see D. König, *Theorie der endlichen und unendlichen Graphen*, Leipzig, 1936, Ch. 14.

[7] H. Whitney, loc. cit., Theorems 12 and 27.