# A REPRESENTATION OF GENERALIZED BOOLEAN RINGS

By N. H. McCoy and Deane Montgomery

1. **Introduction.** Stone has recently shown[1] that every Boolean ring is isomorphic to a ring of subclasses of some class. As Stone himself remarks, there is a close relation between the representation of Boolean rings and the theory of direct sums of rings. The theorem just stated is clearly equivalent to the theorem that every Boolean ring is isomorphic to a subring of a direct sum of rings $F_2$.[2] We present here a simple direct proof of this theorem in a somewhat more general case.

A commutative ring $R_p$ is said to be a *generalized Boolean ring of index* $p$ (often abbreviated *p-ring*) if $p$ is a prime and if for every $a$ in $R_p$ it is true that $a^p = a$ and $pa = 0$. A Boolean ring as defined by Stone is therefore a 2-ring.[3] We show here that a $p$-ring is isomorphic to a subring of a direct sum of rings $F_p$. The interest of this theorem lies partly in its generality and partly in the simplicity of the proof, which is based on an exploitation of a device used by Alexander and by Alexander and Zippin.[4] Inasmuch as our proof, like Stone's, demands the existence of certain homomorphisms and inasmuch as we prove the existence of these homomorphisms by a method analogous to Stone's,[5] our proof makes use of transfinite induction.

2. **Subrings of direct sums.** For the theorem given here on subrings of direct sums neither of the rings considered need be commutative.

Theorem 1. *A necessary and sufficient condition that a ring $R$ be isomorphic to a subring of a direct sum of rings $K$ is that for every $a \neq 0$ in $R$ there is a homomorphism $h$ of $R$ into a subring of $K$ such that $h(a) \neq 0$.*

Consider first the necessity of the condition. Assume then that the elements of $R$ are functions $f$ defined on a certain set $M$ with values in $K$.[6] If $f_1$ in $R$ is not zero, there is some element $m$ such that $f_1(m) \neq 0$. We obtain a homomorphism of $R$ into a subring of $K$ by making correspond to any $f$ in $R$ the value $f(m)$. This homomorphism is not zero on $f_1$ and therefore satisfies the condition of the theorem.

Received January 26, 1937.

[1] M. H. Stone, *The theory of representations for Boolean algebras*, Transactions of the American Mathematical Society, vol. 40 (1936), pp. 37–111. See also Garrett Birkhoff, *On the combination of subalgebras*, Proc. Camb. Phil. Soc., vol. 29 (1933), pp. 441–464.

[2] In general, for any prime $p$, $F_p$ denotes the field of integers reduced modulo $p$.

[3] When $p = 2$, the commutativity and the fact that $pa = 0$ follow from the assumption $a^p = a$.

[4] Annals of Mathematics, vol. 35 (1934), pp. 389–395; vol. 36 (1935), pp. 71–85.

[5] Loc. cit., pp. 102–104.

[6] A direct sum consists of the set of all such functions.