RINGS OF SETS

By Garrett Birkhoff

1. **Definitions.** Following Hausdorff,[1] a family $\mathfrak{F}$ of subsets of a class I is said to form a "ring" if and only if it contains, with any two sets[2] S and T, their *sum* (or union) $S \cup T$ and their *product* (or intersection) $S \cap T$. Clearly a ring contains, with any finite number of subsets $S_1, \cdots, S_n$, their sum $S_1 \cup \cdots \cup S_n$ and their product $S_1 \cap \cdots \cap S_n$.

The family $\mathfrak{F}$ is said to constitute a "complete ring" if and only if it contains, with any subfamily $\mathfrak{S}$ of sets S_α, their sum $\bigvee_{\alpha \in \mathfrak{S}} S_\alpha$ and their product $\bigwedge_{\alpha \in \mathfrak{S}} S_\alpha$. The family $\mathfrak{F}$ is also said to be a "σ-ring" if and only if it contains, with any *countable* subfamily $\mathfrak{S}$ of sets S_α, their sum $\bigvee_{\alpha \in \mathfrak{S}} S_\alpha$ and their product $\bigwedge_{\alpha \in \mathfrak{S}} S_\alpha$.

It is obvious that rings containing only a finite number of sets, and σ-rings containing only a countable number of sets, are necessarily complete rings. These theorems can be improved by using chain conditions; however, the family $\mathfrak{C}$ of all finite sets of integers is a countable ring which is not a σ-ring (and a fortiori not complete), while the family $\mathfrak{D}$ of all countable subsets of the continuum is a σ-ring which is not complete.

2. **The importance of the subject.** Rings of sets are mathematically important for a number of reasons. They are conceptually important because one can define them so simply in terms of two fundamental operations. They are also important because the sets of any class I carried within themselves by any one-valued transformation of I into itself are a complete ring. (The proof of this will be left to the reader.) Also, as is well known, the open and closed subsets of any topological space constitute rings, and the measurable subsets of any Cartesian n-space constitute a σ-ring.

Again, the reader will immediately see that

(2α) The sets common to all the rings (resp. σ-rings or complete rings) of any aggregate of rings of subsets of any class I themselves form a ring (resp. σ-ring or complete ring).

It follows that the closed subsets of any topological space Σ invariant under any group of transformations constitute a ring. The study of these rings is important in dynamics,[3] where, however, the existence of minimal closed and connected constituents introduces special considerations. It follows also that

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[1] *Mengenlehre*, 1927 (2d ed.), p. 77.

[2] We shall systematically use small Latin letters to denote elements, Latin capitals to denote sets of elements, and German capitals to denote families of sets.

[3] Especially in the theory of so-called "central motions". Cf. G. D. Birkhoff, *Dynamical Systems*, 1927, Chap. VII, §6 ff.