ON THE ZEROS OF JACOBI POLYNOMIALS, WITH APPLICATIONS

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Introduction. This paper deals principally with the generalized Jacobi polynomials

$$J_n(x; \alpha, \beta) \equiv J_n(x) \equiv (1 + x)^{1-\alpha}(1 - x)^{1-\beta} \frac{d^n}{dx^n} [(1 + x)^{n+\alpha-1}(1 - x)^{n+\beta-1}]$$

$$\equiv (-1)^n n! \binom{2n + \alpha + \beta - 2}{n} \phi_n(x; \alpha, \beta); \tag{1}$$

$$\phi_n(x; \alpha, \beta) \equiv \phi_n(x) \equiv x^n - S_n x^{n-1} + \cdots \qquad (n = 0, 1, 2, \cdots),$$

defined (except for constant factors) for all real α, β as the polynomial solutions of the differential equation

$$\begin{aligned}(1 - x^2)J_n''(x) + [\alpha - \beta - (\alpha + \beta)x] J_n'(x) \\ + n(n + \alpha + \beta - 1)J_n(x) = 0 \qquad (n = 0, 1, \cdots).\end{aligned} \tag{2}$$

For arbitrary α, β several authors have discussed the number of real zeros of $J_n(x; \alpha, \beta)$. Stieltjes [1][1] gave a method of finding the number of zeros in the intervals $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$ but he stated the result only when $\alpha, \beta > 0$. Shibata [2] gave a table for the number of zeros when they are all real and α, β are not negative integers or zero. Lawton [3] gave complete results for the closed interval $(-1, 1)$ when n is sufficiently large. The results of Hilbert [4], Klein [5], Van Vleck [6], and Hurwitz [7] for the zeros of the hypergeometric function may also be applied to Jacobi polynomials.

Here we find the number of zeros of $J_n(x; \alpha, \beta)$ inside the intervals $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$ and, in addition, at $x = \pm 1$ (from which the number of imaginary zeros is easily obtained) when α, β are arbitrary. The method employed is new and other properties of $J_n(x; \alpha, \beta)$ are developed as well.

In case $\alpha, \beta > 0$, the $J_n(x)$ form, as is known, an orthogonal system

$$\int_{-1}^{1} p(x)J_m(x)J_n(x)dx = 0, \qquad p(x) = (1 + x)^{\alpha-1}(1 - x)^{\beta-1} \tag{3}$$

$$(m \neq n; m, n = 0, 1, \cdots).$$

Received December 7, 1936; presented to the American Mathematical Society, April 20, 1935. The author wishes to express his gratitude to Professor J. Shohat for his valuable suggestions.

[1] The numbers in brackets refer to the bibliography at the end of the paper.