THE MINIMA OF FUNCTIONALS WITH ASSOCIATED SIDE CONDITIONS

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In his doctoral dissertation the author obtained a generalization of the calculus of variations which does not include either the problem of Lagrange with fixed end-points or the more general problem of Bolza. That is to say, the theory therein presented includes only the ordinary problem of the calculus of variations and certain non-calculus of variations problems which have fixed end-points and no side conditions.[1] To remedy this defect a more general situation is considered in the present paper; more specifically it is proposed to find conditions that a functional having certain differentiability properties be a minimum in a class of functions satisfying a system of integro-differential equations and passing through two fixed points.

This problem is transformed into one having only generalized end conditions and is then treated by a method which is a generalization of the technique adopted in the author's thesis. Analogues of the Lagrange multiplier rule, the Clebsch condition, and the Jacobi-Mayer condition are obtained for the transformed problem.

The analogue of the Jacobi-Mayer condition is especially interesting, since the theory of the fixed end-point problem of Lagrange with integro-differential side conditions does not contain such a condition.[2] Moreover this condition does not reduce to the ordinary Jacobi condition for the simple problem of the calculus of variations, as has been shown.[3]

1. **The problems and the transformation.** We shall start with the following problem: to find necessary conditions that an arc

$$\xi_i = \xi_i(s) \qquad (i = 1, \cdots, n; 0 \leqq s \leqq 1)$$

minimize a functional I in the class of arcs satisfying the integro-differential conditions

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[1] *Conditions for a minimum of a functional*, Chicago Doctoral Dissertation (1936); it is expected that this paper will soon appear in the third volume of the *Contributions to the Calculus of Variations* (Chicago). In the subsequent footnotes it will be indicated as Paper I.

[2] See, e.g., L. M. Graves, *A transformation of the problem of Lagrange in the calculus of variations*, Transactions of the American Mathematical Society, vol. 35 (1933), pp. 675–682.

[3] See Paper I, pp. 32–37.