THE EXPANSION THEORY OF ORDINARY DIFFERENTIAL SYSTEMS OF THE FIRST ORDER

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1. **Introduction.** It is a sufficiently curious fact that in the extensive literature of the expansion of arbitrary functions in series of characteristic solutions of ordinary linear differential systems almost no works dealing with the case of the differential systems of the first order are to be found.[1] This is the more remarkable since in this case the integration of the differential system is possible, and much of the analysis which invariably encumbers the discussions of systems of higher order is thereby obviated. To be sure, the Fourier's series, which stands as a prototype in this field, is usually associated with a differential system of the second order, and so the systems of higher order would naturally have suggested themselves as generalizations. Even for purposes of generalization, however, the system of the first order also merits attention, for because of the relative simplicity of its analysis a material generalization becomes possible in the way of a relaxation of restrictive hypotheses. This is found to be far from trivial. The theory of its expansions, as it is to be found in the literature, involves in fact a number of striking peculiarities, in which it contrasts sharply even with the expansion theories of the most closely analogous differential systems of the second order. Thus, by way of instance, one and the same expansion may be generated by an infinity of essentially distinct (i.e., non-equivalent) functions; and again the expansion of a given function, though it may converge, only rarely converges to the function immediately concerned.

It is the purpose of the present paper to present here a new expansion theory for the differential systems of the first order, one which differs from that in the literature and is believed to have advantages over it. It will be found to permit, on the one hand, of a material further relaxation of the restrictions upon the system, and to lead, on the other hand, to results which, on the whole, are much more nearly in consonance with those which obtain in the existing theories for systems of the second or higher orders. In particular, it will be found that the formal association of a function with an expansion is in a suitable sense unique, and that under quite customary conditions an expansion converges to the function with which it is formally associated. Even so, to be sure, some distinctive peculiarities persist. These will generally be recognizable, however, as inherent in the nature of the case.

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[1] A notable exception is M. H. Stone, *An unusual type of expansion problem*, Transactions of the American Mathematical Society, vol. 26 (1924), p. 335.