THE ERGODIC FUNCTION OF BIRKHOFF

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Introduction. In his New Orleans lecture, Professor Birkhoff[1] introduced the concept of the ergodic function $T(\epsilon)$ as the least time T which elapses before the point P of some motion can come within a distance ϵ of every point of the phase space. He is led to conjecture that in the general closed recurrent case possessing no stable periodic motions the ergodic function is of the order of $\epsilon^{-(n-1)}$, n being the number of dimensions of the phase space. The purpose of this paper is to consider the closed, transitive dynamical systems provided by the geodesics on topologically closed surfaces of constant negative curvature. The principal result is the establishment of upper bounds for the ergodic functions of these dynamical systems.

The metric chosen for the phase space is patterned after that used by Morse.[2] The "time" T along a motion is taken as the "H-length" measured along a geodesic on the surface (see §9 of the present paper).

If ϵ be chosen in compliance with (67), (68), we find that the ergodic function $T(\epsilon)$ satisfies an equality of the form

$$T(\epsilon) < \epsilon^{-\omega}\left[A \log \frac{B}{\epsilon} + C\cdot\right],$$

where A, B, C and ω depend on the genus p of the surface as set forth in (77), (78), (79) and (80). All of these constants tend to $+\infty$ for $p \to +\infty$, the constant ω, in particular, being bounded by the inequalities (81), so that $\omega > 4 > 2$ for $p = 2, 3, \cdots$. Since $n - 1 = 2$ in the case under consideration, the order of $T(\epsilon)$ as conjectured by Birkhoff lies well below that of the upper bound we have found.

It may not be amiss to point out that the upper bound we have secured can doubtless be sharpened considerably, for in deriving it we compute an upper bound for the magnitude of the interval of time during which a point P of a certain motion comes within a distance ϵ of every point of the phase space at *least once*, the point P perchance coming *repeatedly* within a distance ϵ of some or all of the points of the phase space. In addition, the upper bound, $(4p)^{2m}$, given in §10 for the number of sets S_M^* can in all probability be improved upon, with a resultant improvement in the upper bound for the ergodic function.

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[1] G. D. Birkhoff, Bull. Amer. Math. Soc., vol. 38 (1932), pp. 375–377.

[2] M. Morse, Jour. de Math., vol. 14 (1935), pp. 52–53. This paper will be referred to as Morse I.