THE CLASSES OF INTEGRAL SETS IN A QUATERNION ALGEBRA

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1. **Introduction.** Let $\mathfrak{A}$ be a rational generalized quaternion algebra with the fundamental number d.[1] A set of integral elements in $\mathfrak{A}$, or more briefly an integral set, is one with certain properties R, C, U, M as defined by Dickson.[2] Two integral sets are said to be equivalent, or of the same type, if there is a one-to-one correspondence between the elements of the sets which is preserved under addition and multiplication. All the sets equivalent to a given set will be said to form a class. Two integral sets, $\mathfrak{G}$ and $\mathfrak{G}_1$, belong to the same class if and only if there is a non-singular element α in $\mathfrak{A}$ such that $\mathfrak{G}_1 = \alpha\mathfrak{G}\alpha^{-1}$.[3]

By a result due to Artin,[4] the number H of classes of integral sets in $\mathfrak{A}$ is equal to the number of classes of equivalent right ideals in an arbitrarily chosen integral set $\mathfrak{G}$, of $\mathfrak{A}$, Artin's definition of equivalent ideals being broader than the usual definition.

The principal purpose of this paper is to show that there is a one-to-one correspondence between the classes of integral sets in $\mathfrak{A}$ and certain classes of ternary quadratic forms. These classes of forms are the non-negative classes or the improperly primitive non-negative classes in a certain genus G, according as d is even or odd. G is uniquely determined by d. If $d < 0$, by a known theorem there is a single class of forms in G and therefore $H = 1$.[5]

We shall also determine a relatively simple basis of an arbitrarily chosen integral set in $\mathfrak{A}$.

2. **A normal basis of an integral set.** If $\lambda_0, \cdots, \lambda_3$ form a basis of an integral set $\mathfrak{G}$, then $\lambda_i\lambda_j = \Sigma_k c_{ijk}\lambda_k$ $(i, j = 0, \cdots, 3)$. A set $\mathfrak{G}_1$ is equivalent to $\mathfrak{G}$ if and only if it has a basis $\xi_0, \cdots, \xi_3$ such that $\xi_i\xi_j = \Sigma_k c_{ijk}\xi_k$ $(i, j = 0, \cdots, 3)$. The following theorem is a consequence of certain results due to Brandt.[6]

Theorem 1. *Let $\mathfrak{A}$ be a generalized quaternion algebra with the fundamental*

Received November 16, 1936.

[1] For the definition of d, see Brandt, *Idealtheorie in Quaternionenalgebren*, Mathematische Annalen, vol. 99 (1929), p. 9.

[2] *Algebras and their Arithmetics*, pp. 141, 2. It may be shown that our definition of an integral set is equivalent to Brandt's definition of a *maximaler Integritätsbereich*, loc. cit., p. 11.

[3] Deuring, *Algebren*, p. 89.

[4] Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, vol. 5 (1927), p. 288, Theorem 20.

[5] In another paper, it was shown that if $d < 0$, then every one-sided ideal in an integral set is principal. (Transactions of the American Mathematical Society, vol. 40 (1936), p. 322.) From this and Artin's result, cited above, it again follows that if $d < 0$, then $H = 1$.

[6] Loc. cit., pp. 8–11.