SUMMABILITY OF DOUBLE FOURIER SERIES

BY J. J. GERGEN

Part I

1.1. Introduction. We consider in Part I the M. Riesz and the Cesàro sums of a simple series

$$(1.11) \qquad \sum_{p=0}^{\infty} a_p ,$$

that is, the sums

$$\sigma_\alpha(x) = \sum_{p<x} (x-p)^\alpha a_p, \qquad S_\alpha(m) = \sum_{p=0}^{m} A_{m-p}^\alpha a_p,$$

where, in the Cesàro sums, A_p^α is the coefficient of x^p in the expansion

$$(1-x)^{-\alpha-1} = 1 + \sum_{p=1}^{\infty} A_p^\alpha x^p.$$

Riesz's theorem that for $0 \leqq \alpha$ the existence of either of the limits

$$\lim_{m\to\infty} S_\alpha(m)/A_m^\alpha = L, \qquad \lim_{x\to\infty} \sigma_\alpha(x)/x^\alpha = L$$

implies that of the other is of course well known.[1] In Theorems I and II below we give some formulas expressing in a simple manner each of the sums in terms of the other. On the basis of these theorems Riesz's result can readily be obtained. The derivations do not in general follow Riesz's procedure; and as regards conciseness there seems to be some advantage in following this second method of approach. Theorems I and II can also be used to advantage to simplify the extensions to two variables of Riesz's theorem given by Dr. S. B. Littauer and the author.[2] In Part II of the present paper we shall apply these theorems in connection with double Fourier series.

THEOREM I. *Let $0 < \alpha$. Let k be the integral part of α. Let $f(x)$ be continuous with its derivatives $f', f'', \cdots, f^{(k+2)}$ for $0 \leqq x$, and satisfy*

$$(1.12) \qquad \begin{aligned} f(0) &= f'(0) = \cdots = f^{(k+1)}(0) = 0, \\ f(x) &= \Gamma(\alpha+x)/\{\Gamma(x)\Gamma^2(\alpha+1)\} \text{ for } 1 \leqq x. \end{aligned}$$

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[1] For a presentation of Riesz's proof see Hobson, *Theory of Functions of a Real Variable*, vol. 2, 1926, pp. 90–98.

[2] J. J. Gergen and S. B. Littauer, *Continuity and summability for double Fourier series*, Transactions of the American Mathematical Society, vol. 38 (1935), pp. 401–435. This paper will be denoted by A.