THE QUADRATIC SUBFIELDS OF A GENERALIZED QUATERNION ALGEBRA

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1. **Introduction.** Let $\mathfrak{A}$ be a rational generalized quaternion algebra with the fundamental number d, as defined by Brandt.[1] Every element of $\mathfrak{A}$, not rational, is a root of a quadratic equation with rational coefficients, and hence defines a quadratic field. The question arises as to what quadratic fields are contained in $\mathfrak{A}$. The purpose of this note is to prove the following

Theorem. *Let $\mathfrak{A}$ be a rational generalized quaternion algebra, with the fundamental number d, and let F be a quadratic field. $\mathfrak{A}$ contains a field equivalent to F if and only if*

(a) *F is imaginary when $d > 0$;*

(b) *no rational prime factor of d is the product of two distinct prime ideals in F.*

Hasse proved a theorem on the splitting fields of an algebra which, when properly specialized, is equivalent to the above theorem, his results being in terms of the p-adic extensions of $\mathfrak{A}$ and of F.[2] Our proof is independent of Hasse's and is short and elementary.

2. **Proof of necessary conditions.** Suppose $\mathfrak{A}$ contains F. Let F be defined by $(-\alpha)^{\frac{1}{2}}$, α being an integer with no square factor > 1. If $d > 0$, by the definition of d, $\mathfrak{A}$ contains no element with a negative norm. Hence F is imaginary.

$\mathfrak{A}$ contains an element i such that $i^2 = -\alpha$. Then the trace, or double the scalar part, of i is zero. It may be shown that $\mathfrak{A}$ also contains a non-singular element j, such that the trace of j and the trace of ij are zero. Then $1, i, j, ij$ are linearly independent, and hence form a basis of $\mathfrak{A}$, $j^2 = -\beta \neq 0$, where β is rational, and $ji = -ij$. We shall assume, without loss of generality, that β is a rational integer with no square factor >1.

Let $\alpha = \alpha_1\delta$, $\beta = \beta_1\delta$, where δ is the positive g.c.d. of α and β. Then $d = \pm AB\Delta$ or $d = \pm 2AB\Delta$, where A, B, Δ are certain positive odd divisors of α, β, δ respectively.[3] By the same reference, d is even if and only if

$$(1) \qquad (\alpha_1 + \beta_1)(\beta_1 + \delta)(\delta + \alpha_1)(\alpha_1 + \beta_1 + \delta) \equiv 8 \pmod{16}.$$

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[1] Brandt, *Idealtheorie in Quaternionenalgebren*, Mathematische Annalen, vol. 99 (1928), p. 9.

[2] Hasse, *Die Struktur der R. Brauerschen algebrenklassengruppe über einem algebraischen Zahlkörper*, Mathematische Annalen, vol. 107 (1933), pp. 731–760; Deuring, *Algebren*, p. 118.

[3] *On the fundamental number of a rational generalized quaternion algebra*, this Journal, vol. 1 (1935), pp. 433–435. This paper will be referred to hereafter as FN.