EQUIVALENCE OF MULTILINEAR FORMS SINGULAR ON ONE INDEX

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1. Introduction. Any p-way matrix $A = (a_{ij\cdots k})$ of order n can be "factored" in the form

$$(1') \qquad A = \left(\sum_{\alpha=1}^{h} a_{\alpha i}\, b_{\alpha j} \cdots d_{\alpha k}\right) \qquad (i, j, \cdots, k = 1, \cdots, n),$$

where $h \leqq n^{p-1}$. Hitchcock,[1] using the polyadic point of view, has determined minimum values of h for some given numerical values of n and p. The representation (1′) implies that *any multilinear form*

$$F = a_{ij\cdots k}\, x_i y_j \cdots z_k \qquad (i, \cdots, k = 1, \cdots, n)$$

(repeated indices indicate summation) *is equivalent under transformations*

$$(2_1) \qquad x'_\alpha = a_{\alpha i} x_i\,,$$

$$(2_2) \qquad y'_\beta = b_{\beta j} y_j\,,$$

$$\cdots \qquad \cdots$$

$$(2_p) \qquad z'_\gamma = d_{\gamma k} z_k\,,$$

to the form

$$R = x'_\alpha y'_\alpha \cdots z'_\alpha \qquad (\alpha = 1, \cdots, h),$$

where $h \leqq n^{p-1}$ and the transformations $(2_1), \cdots, (2_p)$ are not necessarily non-singular.

We shall say that the matrix $(a_{ij\cdots k}\, a_{\alpha i})$ of the form F' obtained from F by applying the transformation $x_i = a_{\alpha i} x'_\alpha$ to F, where $(a_{\alpha i})$ is non-singular, *is equivalent to* $(a_{ij\cdots k})$; we shall also say that F' is equivalent to F. If the 2-way matrices $(a_{\alpha i}), \cdots, (d_{\alpha k})$ of (1′) are all singular on their columns (α being taken as the row index in these matrices), the matrix A is equivalent to a matrix of lower order of the form (1′), where at least one of the matrices $(a_{\alpha i}), \cdots, (d_{\alpha k})$ is non-singular on its columns. The number h of (1′) is then between the limits $n \leqq h \leqq n^{p-1}$. In another paper[2] the author treated the special case where h takes the minimum value n. He obtained necessary and sufficient conditions for the factorability of a matrix A into the form (1′), where the matrices $(a_{\alpha i}), \cdots, (d_{\alpha k})$ are all non-singular. The method of

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[1] F. L. Hitchcock, *A new method in the theory of quantics*, Journal of Mathematics and Physics, vol. 8 (1929), p. 83.

[2] R. Oldenburger, *Non-singular multilinear forms and certain p-way matrix factorizations*, Transactions of the American Mathematical Society, vol. 39 (1936), pp. 422–455. This paper will be denoted by N. S.