# ON CERTAIN EQUATIONS IN RELATIVE-CYCLIC FIELDS

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**1. Introduction.** Let $F$ be a quite arbitrary field—the characteristic may be 0 or some prime $p$. Let $W$ be a field containing $F$ such that $W/F$ is cyclic of relative degree $k$. The group of $W/F$ is generated by the substitution $S$: if $\alpha$ is some quantity in $W$, we shall use the notation $\alpha^S$ to denote the result of operating on $\alpha$ with $S$. If then $\alpha$, $\beta$ are assigned elements of $W$, the equations which we shall study are

$$\xi^S = \alpha\xi \tag{1.1}$$

and

$$\eta^S = \alpha\eta + \beta; \tag{1.2}$$

it is of course supposed that $\xi$ and $\eta$ also are in $W$.

Suppose $W = F(\vartheta)$, that is, $W$ is generated by adjoining $\vartheta$ to $F$, where $\vartheta$ is a root of $f(\vartheta) = 0$, and $f(x)$ is a polynomial with coefficients in $F$ and irreducible in $F$. It is convenient to assume that the coefficient of the highest power of $x$ in $f(x)$ is unity. Let $\alpha = g(\vartheta)$, where $g(x)$ is a polynomial with coefficients in $F$. Then we show that (1.1) has a non-trivial solution if and only if

$$R(g, f) = 1;$$

here $R(g, f)$ is the resultant of the polynomials $g$ and $f$, and may be calculated by means of the division algorithm. If $g$ satisfies certain conditions, a theorem of reciprocity for (1.1) may be stated; in particular, if $F$ is a finite field, this reduces to a known theorem (see §5).

As for equation (1.2), if $\alpha$ is such that (1.1) is not satisfied, then (1.2) has a unique solution. If, however, (1.1) does admit of a non-trivial solution, then we may assume $\alpha = 1$, and our equation becomes

$$\xi^S = \xi + \beta. \tag{1.3}$$

If now we put $\beta = h(\vartheta)$, where $h(x)$ is a properly chosen polynomial in $F$, then we prove that (1.3) is solvable if and only if the coefficient of $x^{k-1}$ in $h(x)\, f'(x)$, reduced modulo $f(x)$, is zero; here $f'(x)$ denotes the derivative of $f(x)$.

In §4 some properties of the solutions of (1.3) are derived. Finally in §5 we assume $F$ to be a finite field and the result for (1.3) as well as for (1.1) is seen to reduce to a known theorem.

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