FUCHSIAN GROUPS AND TRANSITIVE HOROCYCLES

By Gustav A. Hedlund

Let U denote the unit circle in the complex z-plane and let Ψ be its interior. The metric

$$ds^2 = \frac{4 \mid dz \mid^2}{(1 - z\bar{z})^2}, \qquad z\bar{z} < 1 \tag{0.1}$$

defines a hyperbolic geometry in Ψ, the geodesics or *hyperbolic lines* of which are arcs of circles orthogonal to U. These hyperbolic lines will be designated as *H-lines*. The *hyperbolic distance* between two points of Ψ is defined as $\int ds$, where ds is given by (0.1) and the path of integration is the H-line segment joining these two points. The hyperbolic distance between P_1 and P_2 will be denoted by $H(P_1, P_2)$. The metric (0.1) is invariant under linear fractional transformations taking U into itself and Ψ into itself and these transformations transform hyperbolic lines into hyperbolic lines. Hence hyperbolic distance is invariant under all such transformations and these are the rigid motions of the geometry under consideration.

The curves in Ψ of constant geodesic curvature fall into four groups according to their geometrical properties (see e.g. Carathéodory,[1] pp. 22–25). If we denote geodesic curvature by g_c, these classes are as follows:

Class 1. $g_c = 0$. These are the H-lines and are arcs of circles orthogonal to U.

Class 2. $0 < g_c < 1$. These are the *hypercycles.* They are arcs of euclidean circles each of which meets U in two distinct points. The angle at which these curves meet U is uniquely determined by g_c and assumes all values between 0 and $\frac{1}{2}\pi$. The hypercycles are equidistant curves. That is, all the points of any given one are equidistant, in the hyperbolic sense, from the H-line which has the same end points on U.

Class 3. $g_c = 1$. These are the *horocycles* (oricycles). They are euclidean circles which are internally tangent to U.

Class 4. $g_c > 1$. These are the *hyperbolic circles* and lie entirely interior to U. All points of any given one are at the same H-distance from a fixed point in Ψ. These hyperbolic circles are also euclidean circles.

Let F be a fuchsian group with U as principal circle. If points congruent under F are considered identical, a two-dimensional manifold M, of constant negative curvature, is defined. The combinatorial topological properties of M are determined by F. If, in particular, F has a fundamental region lying, together with its boundary, in Ψ and F contains no elliptic transformations,

Received March 31, 1936.

[1] The references are to the bibliography at the end of the paper.