THE NULL DIVISORS OF LINEAR RECURRING SERIES

By Morgan Ward

1. Let

$$(u) \qquad u_0, u_1, u_2, \cdots, u_n, \cdots$$

be a particular solution of the difference equation

$$\Omega_{n+k} = c_1\Omega_{n+k-1} + \cdots + c_k\Omega_n, \tag{1.1}$$

where $u_0, \cdots, u_{k-1}, c_1, \cdots, c_k$ are given rational integers and $c_k \neq 0$. If all of the terms of (u) beyond a certain point are divisible by a given integer m, then m will be said to be a *null divisor* of (u), and (u) a *null sequence* modulo m. In this case there is an integer ν called the *numeric* of (u) modulo m such that[1]

$$u_n \equiv 0 \pmod{m}, \qquad n \geqq \nu, \qquad u_{\nu-1} \not\equiv 0 \pmod{m}.$$

In a previous paper,[2] I have solved the problem of determining the numeric of (u) given its k initial values, the recurrence (1.1) and the null divisor m. In this paper I propose to determine all of the null divisors of (u).[3]

If a and b are null divisors, then ab is also a null divisor provided a and b are co-prime. It suffices then to consider only the case when m is a power of a prime. If p is a prime null divisor of (u), the exponent of the highest power of p dividing all terms of (u) with large suffixes will be called the index of p in (u). If, for example, from a certain point on all terms of (u) are divisible by p^2 but not by p^3, p is of index two.

2. My main results are summarized in the following two theorems.

Theorem 1. *If in the difference equation* (1.1) *we have*

$$c_k \equiv c_{k-1} \equiv \cdots \equiv c_{k-s+1} \equiv 0 \pmod{p}, \qquad c_{k-s} \not\equiv 0 \pmod{p}, \tag{2.1}$$

where p is a prime, and if d_n denotes the greatest common divisor of the $k - s$ consecutive terms

$$u_{n+s}, u_{n+s+1}, \cdots, u_{n+k-1} \qquad (n \geqq 0)$$

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[1] We exclude from consideration the trivial common divisors of $u_0, u_1, \cdots, u_{k-1}$.

[2] *The arithmetical theory of linear recurring series*, Trans. Am. Math. Soc., vol. 35 (1933), pp. 600–628. This paper will be cited here as Theory.

[3] The present paper is a condensation and completion of some earlier results on the same subject. Cf. Abstract 22, Bulletin Am. Math. Soc., vol. 41 (1935), p. 24. Very recently, Abstract 11, Bulletin Am. Math. Soc., vol. 42 (1936), p. 25, Mr. Marshall Hall has given some results on null divisors which we shall discuss in the course of the paper.