BLOCH FUNCTIONS

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In this paper we prove the following theorem.

If $f(x) = x + \cdots$ is regular in $|x| < 1$, and maps $|x| < 1$ on a (many-sheeted) region such that the upper bound of the radii of circles contained in a single sheet of the region is as small as possible, then the unit circle is a natural boundary for $f(x)$.

In proving this, we introduce a method which can probably be used to obtain much more extended results about the functions which map $|x| < 1$ on regions not containing circles any larger than necessary.

This paper is divided into three sections. §1 contains some preliminary material concerning Bloch's Theorem. §2 contains some lemmas about special mapping functions. §3 contains the above theorem and another similar theorem, and some remarks concerning further results about the functions mentioned above.

1. Let R be a region in the complex plane, and let $f(x)$ be regular in R. Then

(1) $f(x)$ is said to be univalent (= schlicht) in R, if $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$, x_1 and x_2 in R.

(2) If S is a point set in the complex plane, $f(x)$ is said to assume S in R, if for every y in S there is an x in R, such that $f(x) = y$.

(3) If S is a point set in the complex plane, $f(x)$ is said to assume S univalently in R, if $f(x)$ is univalent in a subregion R_1 of R, and assumes S in R_1.

Bloch's theorem may be stated in the following form.

If $f(x) = x + \cdots$ is regular in $|x| < 1$, there is a complex number y_0 such that $f(x)$ assumes $|y - y_0| < P$ univalently, where $P > 0$ is an absolute constant.

Here y_0 depends on the function $f(x)$, but P does not.

There are three constants connected with this theorem defined as follows. $\mathfrak{B}$ is the upper bound of constants P which satisfy the theorem. $\mathfrak{L}$ is the upper bound of P if we strike out the word "univalently". $\mathfrak{A}$ is the upper bound of P if $f(x)$ is assumed to be univalent; here it is immaterial whether the word "univalently" is present or not.

The relations $\mathfrak{B} \leqq \mathfrak{L} \leqq \mathfrak{A}$ are obvious. Landau[1] has given numerical bounds for the values of the three constants; in particular, he has shown that $\mathfrak{L} < \mathfrak{A}$. An improved upper bound for $\mathfrak{A}$ was given by me.[2]

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[1] E. Landau, *Über die Blochsche Konstante* $\cdots$, Math. Zeitschrift, vol. 30 (1929), pp. 608–634.

[2] Robinson, *The Bloch Constant* $\mathfrak{A}$ $\cdots$, Bull. Amer. Math. Soc., vol. 41 (1935), pp. 535–540.