FORMAL PROPERTIES OF ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

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1. Construction and properties of symmetry of systems of orthogonal polynomials. The theory of orthogonal polynomials in two variables, as might be anticipated, presents numerous analogies with the corresponding theory in one variable, together with extensive and fundamental differences and complications, which add materially to the interest of the problem, and at the same time limit the scope of an elementary treatment of it.[1]

The "Schmidt process of orthogonalization" is applicable to functions of an arbitrary number of variables. If $\varphi_0(x, y)$, $\varphi_1(x, y)$, $\varphi_2(x, y)$, $\cdots$ form a set of functions integrable with their squares over a region R with no relation of linear dependence connecting any finite number of them (either identically or almost everywhere), it is possible to form a normalized orthogonal sequence $\Phi_0(x, y)$, $\Phi_1(x, y)$, $\Phi_2(x, y)$, $\cdots$ in which Φ_n is a linear combination of $\varphi_0, \varphi_1, \cdots, \varphi_n$. In particular, if R is finite and if $\rho(x, y)$ is a non-negative integrable function having a positive integral over R, application of the process to the linearly independent functions $\rho^{\frac{1}{2}}$, $\rho^{\frac{1}{2}}x$, $\rho^{\frac{1}{2}}y$, $\rho^{\frac{1}{2}}x^2$, $\rho^{\frac{1}{2}}xy$, $\rho^{\frac{1}{2}}y^2$, $\cdots$, taken in this order, gives a sequence of polynomials $q_{nm}(x, y)$, $n = 0, 1, 2, \cdots$; $m = 0, 1, \cdots, n$, such that

$$\int\int_R \rho(x, y) q_{kl}(x, y) q_{nm}(x, y)\, dx\, dy = 0, \qquad |\, n - k\,| + |\, m - l\,| \neq 0,$$

$$\int\int_R \rho(x, y)\, [q_{nm}(x, y)]^2 dx\, dy = 1.$$

The $n + 1$ polynomials $q_{n0}, q_{n1}, \cdots, q_{nn}$ are of the n-th degree in the two variables together, and with respect to ρ as weight function they are orthogonal

Received February 6, 1936; presented to the American Mathematical Society January 1, 1936.

[1] I am indebted to Professor Shohat for the following bibliographical indications: J. Shohat, *Théorie générale des polynomes orthogonaux de Tchebichef*, Mémorial des Sciences Mathématiques, No. 66, Paris, 1934, pp. 20–22, and references 25, 28, 29; F. Didon, *Étude de certaines fonctions analogues aux fonctions X_n de Legendre*, etc., Annales del 'École Normale Supérieure, vol. 5 (1868), pp. 229–310; F. Didon, *Développements sur certaines séries de polynomes*, ibid., vol. 7 (1870), pp. 247–268, and other articles by the same author in vols. 6 and 7 of the same Annales; P. Appell, *Sur une classe de polynomes à deux variables et le calcul approché des intégrales doubles*, Annales de la Faculté de Toulouse, vol. 4 (1890), pp. H 1–20; P. Appell, *Sur les fonctions hypergéométriques de plusieurs variables, les polynomes d'Hermite, et autres fonctions sphériques dans l'hyperespace*, Mémorial des Sciences Mathématiques, No. 3, Paris, 1925; P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques—Polynomes d'Hermite*, Paris, 1926.