CLASSES OF MAXIMUM NUMBERS ASSOCIATED WITH CERTAIN SYMMETRIC EQUATIONS IN n RECIPROCALS. III

By H. A. Simmons and W. E. Block

1. **Introduction.** By extending considerably the methods used by Simmons in the first[1] paper I, and by Stelford and Simmons in the second[2] paper II, we shall obtain results that include as special cases all theorems of I, II (cf. the definition of *remarkable properties* in this section and[3] Theorems 5, 8, 9 and 12). We shall explain in more detail what we do in this paper after we recall from I a few definitions that we use here.

If a solution $x \equiv (x_1, \cdots, x_n)$ of any given equation with which we deal is obtained by Kellogg's process[4] of minimizing the variables $x_1, \cdots, x_{n-1}$ in this order, one at a time, we shall denote it by w and call it the *Kellogg solution* of the given equation. For the equations that we consider the Kellogg solution is (except in §14) one in positive integers. It always belongs to the general class of solutions that we admit, namely, that in which $x_1, \cdots, x_{n-1}$ are positive integers and $x_1 \leqq x_2 \leqq \cdots \leqq x_n$. These solutions include all positive integral solutions and are called *E-solutions* (for extended solutions, beyond those in positive integers). Thus, for a very simple example, the Kellogg solution of $x_1^{-1} + x_2^{-1} = 2/7$ is $x = w = (4, 28)$ and its E-solutions are $(4, 28)$, $(5, 35/3)$, $(6, 42/5)$, and $(7, 7)$. From Theorem 2, p. 887, of I, we know that 28 $(=w_2)$ is the largest number that exists in any E-solution of the given equation and that 28 appears in no E-solution of this equation except w. Furthermore, if $P(x_1, x_2) \equiv P(x)$ is any symmetric polynomial in x_1, x_2 with no negative coefficient, and if $P(x)$ is not a mere constant, Theorem 3 of I contains the following statement as a very special fact: if $x = X$ is any E-solution of the equation $x_1^{-1} + x_2^{-1} = 2/7$ other than its Kellogg solution w, then $P(X) < P(w)$.

Where nothing is said to the contrary, we adopt generally the definitions and notation of I, II. Thus $P(x)$ stands for a polynomial of the type defined above except that $P(x)$ contains n variables instead of 2; and with $i \geqq 0$ and j equal to

Received August 12, 1935; presented to the American Mathematical Society, April 19, 1935. The results of Parts 1, 2, 3 of this paper are due to Simmons; those of Part 4, chiefly to Block, a student at Northwestern University.

[1] Cf. Trans. Amer. Math. Soc., vol. 34 (1932), pp. 876–907.

[2] Cf. Bulletin Amer. Math. Soc., vol. 40 (1934), pp. 884–894.

[3] A theorem has the same number as the section which contains it.

[4] Concerning Kellogg's diophantine problem and extensions of it, cf. O. D. Kellogg, American Mathematical Monthly, vol. 28 (1921), p. 300; D. R. Curtiss, ibid., vol. 29 (1922), pp. 381–387; and Tanzô Takenouchi, Proceedings of the Physico-Mathematical Society of Japan, (3), vol. 3, pp. 78–92.