THE ZEROS OF JACOBI AND RELATED POLYNOMIALS

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Introduction

1. **Definitions.** The ultraspherical polynomials of degree n, $P_n^{(\lambda)}(\cos\vartheta)$, are defined as polynomials not vanishing identically for which the differential equation

$$y'' + \{(n+\lambda)^2 + \lambda(1-\lambda)\sin^{-2}\vartheta\}y = 0 \tag{1}$$

has the solution $y = \sin^\lambda\vartheta \cdot P_n^{(\lambda)}(\cos\vartheta)$. It will also be convenient to consider the generating function of these polynomials normalized in a proper way, namely,

$$(1 - 2w\cos\vartheta + w^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(\cos\vartheta)\cdot w^n. \tag{2}$$

The Jacobi polynomials of degree n, $P_n^{(\alpha,\beta)}(\cos\vartheta)$, are defined as polynomials not vanishing identically for which the differential equation

$$y'' + \left\{\left(n + \frac{\alpha+\beta+1}{2}\right)^2 + \frac{\frac{1}{4}-\alpha^2}{4\sin^2\theta/2} + \frac{\frac{1}{4}-\beta^2}{4\cos^2\theta/2}\right\}y = 0 \tag{3}$$

has the solution $y = [\sin(\vartheta/2)]^{\alpha+\frac{1}{2}}[\cos(\vartheta/2)]^{\beta+\frac{1}{2}}\cdot P_n^{(\alpha,\beta)}(\cos\vartheta)$.

The Jacobi polynomials reduce to the ultraspherical polynomials if $\alpha = \beta = \lambda - \frac{1}{2}$. The ultraspherical polynomials reduce to the Legendre polynomials if $\lambda = \frac{1}{2}$. Concerning further properties of these polynomials, we refer[1] to [**5**] and [**8**].

2. **Previous estimates.** For $\lambda > -\frac{1}{2}$ all of the zeros of the ultraspherical polynomials are real. Let ϑ_k denote the k-th zero in increasing order, $0 < \vartheta_k < \pi$. The following estimates for ϑ_k have been given:

(1) Bruns [**2**] for Legendre polynomials:

$$\frac{k-\frac{1}{2}}{n+\frac{1}{2}}\pi < \vartheta_k < \frac{k}{n+\frac{1}{2}}\pi \qquad (k = 1, 2, \cdots, n). \tag{A}$$

(2) A. Markoff [**4**] and Stieltjes [**7**] for Legendre polynomials:

$$\frac{k-\frac{1}{2}}{n}\pi < \vartheta_k < \frac{k}{n+1}\pi \qquad \left(k = 1, 2, \cdots, \left[\frac{n}{2}\right]\right). \tag{B}$$

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[1] Numbers in bold face type refer to the bibliography at the end of this paper.