CRITICAL POINT THEORY UNDER GENERAL BOUNDARY CONDITIONS

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1. **Introduction.** M. Morse has previously treated the theory of the critical points of a function of class C^2, whose critical values are isolated and the neighborhoods of whose critical sets admit a special type of deformation, ref. [7, 8]. In the present paper it is assumed merely that the function f has continuous first partial derivatives which satisfy Lipschitz conditions and that the critical values (not critical points) of f are isolated. In spite of the fact that the critical sets may *not be locally connected or possess neighborhoods which are contractible,* it is shown that the type numbers of the critical sets are *finite* and depend on the definition of f only in an arbitrarily small neighborhood of the critical set. We emphasize the fact that this part of the treatment *does not depend at all upon the definition of f in the large.*

The authors have also taken up the theory of critical points on an abstract metric space, ref. [10]. The present treatment, although less general, is simpler and more suitable for most applications in analysis. The reader may also refer to the papers of A. B. Brown [2], Lefschetz [5], and Birkhoff and Hestenes [1]. See [12] for a later abstract by Morse.

The second part of this paper contains the first treatment under general boundary conditions. The principal theorem here was announced by Morse in [6]. This part of the paper has important applications in the theory of harmonic functions, as will appear in a subsequent paper by Morse.

Finally, various group theory aspects of the problem are brought out.

I. Type numbers

2. **The function and the critical set.** Let $(x) = (x_1, \cdots, x_n)$ be the rectangular coördinates of a point in euclidean n-space. Let R be a limited, open, n-dimensional region of this space. Let $f(x)$ be a real, single-valued function of[1] class C^1L defined on R. A point of R at which all of the first partial derivatives of f vanish will be called a *critical point* of f. The value of f at a critical point will be called a *critical value* of f. We assume that the critical values of f are isolated. Neighboring each ordinary (non-critical) point of f the differential equations of the trajectories orthogonal to the loci $f =$ constant may be given the form

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[1] A function defined on an open region will be said to be of class C^1L on this region if it is of class C^1 and if its partial derivatives satisfy Lipschitz conditions neighboring each point of the region.