THE IDEAL WARING THEOREM FOR TWELFTH POWERS

By L. E. Dickson

1. **Ideal.** Let g denote the greatest integer $< (3/2)^n$. Let

$$P = g2^n - 1\,, \qquad I = g + 2^n - 2\,. \tag{1}$$

Evidently $P < 3^n$. Consider all the ways in which P can be a sum of positive integral n-th powers, necessarily 1^n or 2^n. There will occur more than I summands except when there are exactly $g-1$ terms 2^n and exactly $2^n - 1$ terms 1. Hence P is a sum of I, but not fewer, n-th powers.

When $n = 2, 3, 4$, $g = 2, 3, 5$, $I = 4, 9, 19$. Lagrange and Euler proved that every positive integer is a sum of four squares. In 1770 Waring conjectured that "every positive integer is a sum of 9 cubes, also of 19 fourth powers, etc." The proof for 9 cubes was first obtained by Wieferich.[1] For fourth powers, the best result yet proved is that 35 suffice, while all integers $< 10^{26}$ are sums of 19. All tables extant confirm the following amplification of Waring's conjecture: *Every positive integer is a sum of* $I = g + 2^n - 2$ *integral n-th powers* $\geqq 0$. This will be called the ideal Waring theorem. Proof has been published only for $n = 2$ and $n = 3$.

2. **Summary for twelfth powers.** We here prove the ideal Waring theorem for $n = 12$, viz.,

Theorem 1. *Every positive integer is a sum of* $I = 4223$ *integral twelfth powers.*

But we go further and prove

Theorem 2. *Every integer* $> 2\cdot 3^{12}$ *is a sum of* 2405 *twelfth powers. Every one* $> 3\cdot 3^{12}$ *is a sum of* 1560 *powers. Every one* $> 2\cdot 5^{12} + 7^{12} + 8^{12}$ *is a sum of* 440.

There are only two earlier results.[2] From Kempner's identity, we find that 6 1/4 billion twelfth powers suffice. By a short table, the writer proved also that 10711 suffice. The present table gives decompositions of 6 908 733 consecutive integers into twelfth powers.

3. **Minimum decompositions.** We employ

$$a = 2^{12} = 4096\,, \qquad b = 3^{12} = 531\,441\,, \qquad c = 4^{12} = 16\,777\,216\,,$$

$$d = 5^{12} = 244\,140\,625\,, \qquad f = 6^{12} = 2\,176\,782\,336\,, \tag{1}$$

$$g = 7^{12} = 13\,841\,287\,201\,, \qquad h = 8^{12} = 68\,719\,476\,736\,,$$

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[1] Math. Annalen, vol. 66 (1909), pp. 99–101. The proof was corrected and greatly simplified by Dickson, Trans. Amer. Math. Soc., vol. 30 (1927), pp. 1–7.

[2] Dickson, Bull. Amer. Math. Soc., vol. 39 (1933), p. 713.