A PARTICULAR SEQUENCE OF STEP FUNCTIONS

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1. **Introduction.** If a sequence of real functions $f_n(t)$ summable on (0, 1) converges in measure to $f(t)$ and if

$$\lim_{n=\infty} \int_\delta f_n(t)\, dt \text{ exists} \tag{1}$$

for every measurable subset δ of (0, 1), then $f(t)$ is summable and

$$\lim_{n=\infty} \int_\delta f_n(t)\, dt = \int_\delta f(t)\, dt$$

uniformly with respect[1] to δ. But (1) may hold for every interval δ in (0, 1) and the conclusion in the weaker form

$$\lim_{n=\infty} \int_0^x f_n(t)\, dt = \int_0^x f(t)\, dt \text{ almost everywhere}$$

may not be true even if it is assumed that $f(t)$ is summable and that

$$f_n(t) = f(t), \text{ (except on a set whose measure approaches zero with } 1/n). \tag{2}$$

The present note is concerned with the behavior of $\int_0^x f_n(t)\, dt$ under the assumption (2) and we assume without loss of generality that $f(t) = 0$. The result is that *there exists a sequence of positive step functions $f_n(t)$ satisfying* (2) *with $f(t) = 0$ such that for every summable function $g(t)$ except for those in a certain set of the first category the sequence $\int_0^x f_n(t)g(t)\, dt$ is everywhere dense in the space of measurable functions.* This is embodied in Theorem 2. The principle (which is Theorem 1) underlying the construction of the sequence $f_n(t)$ is a generalization of an abstraction of an argument used by J. Marcinkiewicz[2] to show the existence of a continuous function $\varphi(t)$ which depends only upon a given sequence of positive numbers $h_n \to 0$ such that an arbitrary measurable func-

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[1] This can be proved by combining results found in the following references. Saks, *Addition to the note on some functionals*, Trans. Amer. Math. Soc., vol. 35 (1933), p. 969; Jeffery, *The integrability of a sequence of functions*, Trans. Amer. Math. Soc., vol. 33 (1931), p. 435, B; and Dunford, *Integration in general analysis*, Trans. Amer. Math. Soc., vol. 37 (1935), p. 447, Theorem 9.

[2] *Sur les nombres dérivés*, Fund. Math., vol. 24 (1935), pp. 305–308.