CONCERNING THE TRANSITIVE PROPERTIES OF GEODESICS ON A RATIONAL POLYHEDRON

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This paper considers geodesics on ordinary polyhedrons[1] in an abstract space. A geodesic on an ordinary polyhedron becomes an ordinary straight line if the sequence of faces belonging to that geodesic is thought of as spread out on a plane. We shall be concerned in what follows only with *rational polyhedrons*, that is, ordinary polyhedrons in which the sum of all angles at any corner is a rational multiple of π. The problem may be considered as an elementary illustration of the 'billard ball' problem considered by Birkhoff in Chapter VI of his Colloquium Publication *Dynamical Systems* and was suggested to us by Wintner. The geometrical condition of rationality defined above is, in the main, the condition on integrability in the sense of Birkhoff or, in the case of a periodic solution, the rationality of the rotation number.

If a direction is moved parallel to itself along any closed curve, which meets no corners, it can only come back to a finite number of positions, for the closed curve can be deformed continuously, without passing over any corners, into another which admits a decomposition into simple loops, each loop consisting of a closed circuit about a corner. Each circuit changes the direction by the sum of the angles about the corner and there is but a finite number of corners.

Now[2] an "Ueberlagerungsfläche" P for the rational polyhedron Π may be defined as follows. Consider an arbitrary but fixed direction[3] on one of the faces of Π and all possible simple curves on Π starting from a fixed point in the interior of that face and not meeting any corners. Let the initial direction be moved parallel to itself along each of these curves. If two such curves have a common end point but different directions there, we consider the two end points to be on different faces. The totality of faces, distinct in this sense, constitutes the *Ueberlagerungsfläche* P. The most important properties of P are the following.

(1) P is a finite polyhedron. For to each face of Π there corresponds a finite number of faces of P.

Received by the Editors of the Annals of Mathematics November 9, 1934, accepted by them, and later transferred to this journal.

[1] Ordinary polyhedrons are meant in the sense of E. Steinitz, *Polyeder und Raumeinteilungen*, Encyklopädie der Mathematischen Wissenschaften, vol. III, Part I_2, p. 15. Geodesics on polyhedrons have been considered, for instance, by P. Stäckel, *Geodätische Linien auf Polyederflächen*, Rend. Circ. Mat. Palermo, vol. 22 (1906), pp. 141–151 and by C. Rodenberg, *Geodätische Linien auf Polyederflächen*, ibid., vol. 23 (1907), pp. 107–125.

[2] For the ideas involved here cf. H. Weyl, *Die Idee der Riemannschen Fläche*, Berlin, 1923.

[3] The particular choice of this direction has, of course, no influence on the construction we are going to make.