CONVEX POLYHEDRA AND CRITERIA FOR IRREDUCIBILITY

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1. **Introduction.** This paper gives an application of Minkowski's[1] theory of convex polyhedra to the construction of irreducibility criteria for polynomials in several variables, thus generalizing the results of Dumas[2] for polynomials in one variable obtained by the use of convex polygons. The results of Minkowski referred to concern the nature of the least convex polyhedron determined by the set of points $\{p\}$ where each point p is of the form $\sum s_i p_i$, each s_i a constant > 0 and each p_i a point of a given polyhedron K_i. As a special case, these results include the case of the polyhedron K determined by two given polyhedra K_1 and K_2 with $s_1 = s_2 = 1$. This is the case which is of interest to us for the results that follow. For reasons obvious later, we shall term K the product, rather than the sum, of K_1 and K_2 and shall denote this relation by the notation $K = K_1 \cdot K_2$.

2. **Decomposability.** If now we consider the converse problem, namely, given K to determine K_1 and K_2 such that $K_1 \cdot K_2 = K$, we find first of all that we must assume we are dealing with polyhedra whose vertices have integral coördinates, for otherwise K_1 and K_2 can be chosen in an infinity of ways, i.e., the problem has no meaning. We likewise impose the further restriction that neither of the factors of K shall be a point, since this would simply amount to a translation of K with no accompanying change in its shape. If, under these conditions, it is possible to determine two polyhedra K_1 and K_2 (either or both may be lines or polygons) such that $K_1 \cdot K_2 = K$, we say that K is decomposable. We proceed to set up necessary conditions for the decomposability of K by considering its projections on the coördinate planes.

First projecting the vertices of K on the xy-plane, we determine the least convex polygon containing this set of points. This polygon we name the xy-boundary polygon of K and denote by b_{xy}. We have at once the result that for K to decompose it is necessary that each of the three boundary polygons of K decompose.[3] For if K decomposes into K_1 and K_2, the product of the xy- (or

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[1] H. Minkowski, *Gesammelte Abhandlungen*, vol. 2, pp. 131–229.

[2] G. Dumas, *Sur quelques cas d'irréductibilité des polynômes à coefficients rationnels*, Journal de Mathématiques Pures et Appliquées, (6), vol. 2 (1906), pp. 191–258.

[3] By extending Dumas' result in the plane to cover the entire least convex polygon, it is possible to show that the decomposition of a plane polygon amounts to the division of the sides of the given polygon into two sets such that the sides of each set, translated, wherever necessary, form a closed convex polygon, each side keeping its outer normal unchanged.